

SEMISOLIDITY AND LOCALLY WEAK QUASISYMMETRY OF HOMEOMORPHISMS IN METRIC SPACES

MANZI HUANG, ANTTI RASILA, XIANTAO WANG*, AND QINGSHAN ZHOU

ABSTRACT. In this paper, we investigate the relationship between semisolidity and locally weak quasisymmetry of homeomorphisms in quasiconvex and complete metric spaces. Our main objectives are to (1) generalize the main result in [14] together with other related results, and (2) give a complete answer to the open problem given in [14]. As an application, we prove that the composition of two locally weakly quasisymmetric mappings is a locally weakly quasisymmetric mapping and that it is quasiconformal.

1. INTRODUCTION AND MAIN RESULTS

The quasihyperbolic metric was introduced by Gehring and his students Palka and Osgood in the 1970's [7, 8] in the setting of Euclidean spaces \mathbb{R}^n ($n \geq 2$). Since its first appearance, the quasihyperbolic metric has become an important tool in the geometric function theory of Euclidean spaces.

From late 1980's onwards, Väisälä has developed the theory of (dimension) free quasiconformal mappings (the free theory) in Banach spaces [26, 27, 28, 29, 31], which is based on the quasihyperbolic metric. The main advantage of this approach over generalizations based on the conformal modulus (see [9] and references therein) is that it does not make use of volume integrals, which allows one to study quasiconformality of homeomorphisms in infinite dimensional Banach spaces. In the free theory, Väisälä has mainly studied the relationships between quasiconformal mappings and quasisymmetric mappings, as well as properties of the quasihyperbolic metric and various classes of domains. The importance of the quasihyperbolic metric in this setting arises from the distortion inequality in Definition 1.3 (cf. the Schwarz-Pick type result given in [7]). This line of research has recently attracted substantial interest in the research community (see e.g. [2, 4, 5, 12, 13, 16, 19, 20, 21, 32]).

The class of quasisymmetric mappings on the real axis were first introduced by Beurling and Ahlfors [3], who found a way to obtain a quasiconformal extension of a quasisymmetric self-mapping of the real axis to a self-mapping of the upper half-plane. This concept was later generalized by Tukia and Väisälä, who studied quasisymmetric mappings between metric spaces [22].

File: fqcmetric-15-8-21-ar.tex, printed: 2015-8-24, 2.19

2000 *Mathematics Subject Classification.* Primary: 30C65, 30F45; Secondary: 30C20.

Key words and phrases. Quasihyperbolic metric; length metric; quasiconformality; quasisymmetry; weak quasisymmetry; locally weak quasisymmetry; semisolidity; quasiconvex metric space.

* Corresponding author.

The research was partly supported by NSF of China (No. 11571216).

In 1998, Heinonen and Koskela [11] proved a remarkable result, showing that these two concepts, quasiconformality and quasisymmetry, are quantitatively equivalent in a large class of metric spaces, which includes Euclidean spaces. In Banach spaces, Väisälä also proved the quantitative equivalence among free quasiconformality, quasisymmetry and weak quasisymmetry. See [31, Theorem 7.15].

Another important result in this topic was very recently obtained by Huang and Liu [14]. They used ideas from Väisälä's approach to show that even weakly quasisymmetric mappings preserve, in a suitable way, the quasihyperbolic metric in quasiconvex and complete metric spaces (see Theorem 1.6 in [14]). Also, in [14], the authors asked that whether the converse of Theorem 1.6 holds or not. As the main purpose of this paper, we shall study the main result and the open problem in [14]. In order to introduce the main result and the problem in [14] and state our results, we need some basic definitions.

Throughout this paper, we always assume that X and Y are metric spaces. Following notations and terminology of [10, 11, 14, 24, 31], we begin with the definitions of quasiconformality and quasisymmetry of homeomorphisms.

Definition 1.1. *A homeomorphism f from X to Y is said to be*

- (1) *quasiconformal if there is a constant $H < \infty$ such that*

$$(1.1) \quad \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)} \leq H$$

for all $x \in X$;

- (2) *quasisymmetric if there is a constant $H < \infty$ such that*

$$(1.2) \quad \frac{L_f(x, r)}{l_f(x, r)} \leq H$$

for all $x \in X$ and all $r > 0$,

where $L_f(x, r) = \sup_{|y-x| \leq r} \{|f(y) - f(x)|\}$ and $l_f(x, r) = \inf_{|y-x| \geq r} \{|f(y) - f(x)|\}$.

Here and in what follows, we always use $|x - y|$ to denote the distance between x and y .

Definition 1.2. *A homeomorphism f from X to Y is said to be*

- (1) *η -quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

for each $t > 0$ and for each triplet x, a, b of points in X ;

- (2) *weakly H -quasisymmetric if*

$$|x - a| \leq |x - b| \text{ implies } |f(x) - f(a)| \leq H|f(x) - f(b)|$$

for each triplet x, a, b of points in X .

Remark 1.1. *The following observations are immediate consequences of Definitions 1.1 and 1.2.*

- (1) *The quasisymmetry implies the quasiconformality;*
- (2) *A homeomorphism f from X to Y is quasisymmetric with coefficient H defined by (1.2) if and only if it is weakly H -quasisymmetric;*

- (3) *The quasisymmetry implies the weak quasisymmetry. In general, the converse is not true (cf. [31, Theorem 8.5]). See [17] for the related discussions.*

In [11], Heinonen and Koskela proved that, for $Q > 1$, quasiconformal mappings between Ahlfors Q -regular metric measure spaces, are quasisymmetric, provided that the source is a Loewner space and the target space satisfies a quantitative connectivity condition. See also [15] for a discussion on the quasisymmetry of quasiconformal mappings in \mathbb{R}^n .

Definition 1.3. *Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains (open and connected), and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. We say that a homeomorphism $f : G \rightarrow G'$ is φ -semisolid if*

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y))$$

for all $x, y \in G$ (see Section 2 for the definition of the quasihyperbolic metric).

In [7], Gehring and Osgood proved that every K -quasiconformal mapping f in a domain $G \subsetneq \mathbb{R}^n$ is a φ -semisolid mapping, where $\varphi(t) = c \max\{t, t^{1/(1-n)}\}$ and $c = c(K, n)$ which means that the constant c depends only on the coefficient K of quasiconformality of f and the dimension n of the Euclidean space \mathbb{R}^n (see [7, Theorem 3]), and thus f^{-1} is also φ -semisolid since it is K -quasiconformal. In [23, Theorem 6.12], Tukia and Väisälä proved that the converse is also true. This implies that f is K -quasiconformal if and only if both f and f^{-1} are φ -semisolid, where K and φ depend on each other and n . This means, by using the terminology in Väisälä's free theory, that quasiconformality and free quasiconformality in \mathbb{R}^n are quantitatively equivalent.

We introduce the following definition, which is a generalization of convexity:

Definition 1.4. *For $c \geq 1$, a metric space X is c -quasiconvex if each pair of points $x, y \in X$ can be joined by an curve γ with length $\ell(\gamma) \leq c|x - y|$.*

We remark that X is convex if and only if $c = 1$, and obviously, if X is c_1 -quasiconvex, then it must be c_2 -quasiconvex for any $c_2 \geq c_1$.

Now, we are ready to state the main result of [14], which is the following.

Theorem A. ([14, Theorem 1.6]) *Let X be a c -quasiconvex and complete metric space and Y a c' -quasiconvex metric space. Suppose $f : G \rightarrow G'$ is weakly H -quasisymmetric, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains. Then there exists a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid, where $\varphi = \varphi_{c,c',H}$.*

An open problem given in [14] can be expressed as follows: Let X be a c -quasiconvex and complete metric space and let Y be a c' -quasiconvex metric space. Suppose that $f : G \rightarrow G'$ is homeomorphic, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains and that there exists a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in G . Is f weakly H -quasisymmetric from G onto G' , where $H = H(c, c', \varphi)$?

The following example shows that the answer to this problem is negative.

Example 1.1. Let $G = \mathbb{C} \setminus \{0\}$ and $f : G \rightarrow G$ be the following inversion:

$$f(z) = \frac{z}{|z|^2}.$$

Then the following statements hold.

- (1) f is 1-semisolid in every subdomain of G ;
- (2) f is not weakly H -quasisymmetric for any $H \geq 1$.

Here and thereafter, except when stated otherwise, G itself is always regarded as a subdomain of G . From the proof of Theorem A in [14], we know that Theorem A also holds in every subdomain of G . Therefore, the problem in [14] can also be reformulated in the following form.

Problem 1. Let X be a c -quasiconvex and complete metric space and let Y be a c' -quasiconvex metric space. Suppose that $f : G \rightarrow G'$ is homeomorphic, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains and that there exists a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in every subdomain of G . Is f weakly H -quasisymmetric, where $H = H(c, c', \varphi)$?

The authors of [14] studied Problem 1 themselves and obtained the following result.

Theorem B. ([14, Theorem 1.8]) Let X be a c -quasiconvex and complete metric space, $G \subsetneq X$ a non-cut-point domain, and $G' \subsetneq Y$ a domain. Suppose $f : G \rightarrow G'$ is a homeomorphism. If for any subdomain $D \subset G$ and for any x and $y \in D$,

$$k_{f(D)}(f(x), f(y)) \leq \varphi(k_D(x, y)),$$

where φ is an increasing function, then f is a K -quasiconformal mapping in G with $K = K(c, \varphi)$.

Again, it follows from Example 1.1 that the answer to Problem 1 is negative. In the following, we shall further develop the ideas of Theorem A and Problem 1. In order to state our main result of this paper, two more concepts are required.

Definition 1.5. Let $G \subsetneq X$ and $G' \subsetneq Y$ be domains. A homeomorphism $f : G \rightarrow G'$ is said to be

- (1) q -locally η -quasisymmetric for some $0 < q < 1$ if there is a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

for every $t > 0$, for each triplet x, a, b of points in $\mathbb{B}^G(z, q\delta_G(z))$ (see Section 3 for the definition) and for all $z \in G$;

- (2) q -locally weakly H -quasisymmetric for some $0 < q < 1$ if

$$|x - a| \leq |x - b| \text{ implies } |f(x) - f(a)| \leq H|f(x) - f(b)|$$

for each triplet x, a, b of points in $\mathbb{B}^G(z, q\delta_G(z))$ and for all $z \in G$.

Remark 1.2. (1) The weak quasisymmetry implies the locally weak quasisymmetry;

- (2) *Both the locally weak quasisymmetry and the weak quasisymmetry imply the quasiconformality.*

Now, we are ready to state our main result.

Theorem 1. *Suppose that X is a c -quasiconvex and complete metric space and Y is a c' -quasiconvex metric space and that $f : G \rightarrow G'$ is homeomorphic, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains. Then the following statements are equivalent:*

- (1) *there is a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in every subdomain of G ;*
- (2) *there are constants $\mu > 0$ and $0 < \alpha < 1$ such that f is φ_1 -semisolid in every subdomain of G , where $\varphi_1(t) = \mu \max\{t^\alpha, t\}$;*
- (3) *there are constants $H \geq 1$ and $0 < q < 1$ such that f is q -locally weakly H -quasisymmetric in G ,*

where the constants μ , α , H , q and the control function φ depend on each other and the constants c and c' .

Remark 1.3. (i) *It follows from Remark 1.2 (1) that the implication from (3) to (1) in Theorem 1 is a generalization of Theorem A.*

(ii) *It follows from the discussions in Section 4 that the condition “ Y being c' -quasiconvex” is unnecessary in the proof of the implication from (1) to (3) in Theorem 1. Hence Remark 1.2 (2) shows that the implication from (1) to (3) in Theorem 1 is a generalization of Theorem B. Further, this implication implies that the assumption “ G being non-point-cut” in Theorem B is redundant.*

(iii) *The equivalence between (1) and (3) in Theorem 1 together with Example 1.1 can be regarded as a complete answer to Problem 1.*

(iv) *As we mentioned in the paragraph next to Definition 1.3, the implication from (1) to (2) in Theorem 1 was first discussed by Gehring and Osgood [7, Theorem 3] for the Euclidean case $X = Y = \mathbb{R}^n$ with $\mu = \mu(K, n)$ and $\alpha = K^{\frac{1}{1-n}}$ when f is K -quasiconformal. Later, Anderson, Vamanamurthy and Vuorinen [1, Theorem 1.12] proved a version of this result that avoids an explicit reference to the dimension n , i.e., $\alpha = K^{\frac{1}{1-n}}$, but $\mu = \mu(K)$, i.e. μ is independent of n . In 1999, Väisälä generalized [7, Theorem 3] and [1, Theorem 1.12] to the setting of Banach spaces [31, Theorem 12.3]. Obviously, the implication from (1) to (2) in Theorem 1 is a generalization of [31, Theorem 12.3].*

Naturally, the above considerations lead one the following problem: Under the assumptions of Problem 1, suppose further that there exists a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in G . Are there constants $H \geq 1$ and $0 < q < 1$ such that f is q -locally weakly H -quasisymmetric in G ?

The following example indicates that the answer to this problem is still negative.

Example 1.2. *Let $G = \{z = x + iy \in \mathbb{C} : y > 0\}$ and $f : G \rightarrow G$ be the following homeomorphism:*

$$f(z) = \begin{cases} x + iy, & \text{if } x \leq 0, \\ x + i(x+1)y, & \text{if } x \geq 0, 0 < y \leq 1, \\ x + i(x+y), & \text{if } x \geq 0, y > 1. \end{cases}$$

Then the following statements hold:

- (a) f is φ -semisolid in G , where $\varphi(t) = \sqrt{3}t$;
- (b) f is not ψ -semisolid in every proper subdomain of G for any homeomorphism $\psi : [0, \infty) \rightarrow [0, \infty)$; and
- (c) f is not q -locally weakly H -quasisymmetric in G for any $H \geq 1$ and $0 < q < 1$.

We remark that Example 1.2 was constructed by Väisälä [31, Example 7.4] when he showed the conclusion that semisolidity does not imply solidity.

As a direct consequence of Theorem 1, we see from [11, Theorem 3.13] that the necessity of quasiconformality of homeomorphisms in [21] is also sufficient when the space is complete and the image space is unbounded as the following result illustrated.

Theorem 2. *Suppose that $f : G \rightarrow G'$ is a homeomorphism between two domains $G \subsetneq X$ and $G' \subsetneq Y$ of globally Q -bounded geometry ($Q > 1$), that X is complete, and that Y is unbounded. Then the following statements are equivalent:*

- (1) f is K -quasiconformal;
- (2) There are constants $\mu > 0$ and $0 < \alpha < 1$ such that f is φ_1 -semisolid in every subdomain of G , where $\varphi_1(t) = \mu \max\{t^\alpha, t\}$;
- (3) There are constants $H \geq 1$ and $0 < q < 1$ such that f is q -locally weakly H -quasisymmetric in G ,

where the constants K , H , q , μ and α depend on each other and the data of the spaces X and Y .

It is easy to see that the composition of two φ -quasisymmetric mappings is also a φ -quasisymmetric mapping, see [22, Theorem 2.2]. But, in [22], Tukia and Väisälä have indicated that the composition of two weakly quasisymmetric mappings need not be weakly quasisymmetric. So, it is significant to study whether the composition of two locally weakly quasisymmetric mappings is quasisymmetric or quasiconformal. As another application of Theorem 1, we obtain the following.

Theorem 3. *Suppose that X_i is a c_i -quasiconvex and complete metric space ($i = 1, 2$) and that X_3 are c_3 -quasiconvex metric spaces. For domains $G_i \subsetneq X_i$ ($i = 1, 2, 3$), if $f : G_1 \rightarrow G_2$ is a q_1 -locally weakly H_1 -quasisymmetric mapping and $g : G_2 \rightarrow G_3$ is a q_2 -locally weakly H_2 -quasisymmetric mapping, then*

- (1) *the composition $g \circ f$ is q -locally H -quasisymmetric;*
- (2) *the composition $g \circ f$ is also H -quasiconformal,*

where the constants q and H depend only on $c_1, c_2, c_3, q_1, q_2, H_1$ and H_2 .

The rest of this paper is organized as follows. In Section 2, we shall prove the two examples: Examples 1.1 and 1.2, and some basic results with respect to quasihyperbolic metric and length metric will be shown in Section 3. The next two sections will be devoted to the proof of the equivalence between (1) and (3) in Theorem 1. The proof of the equivalence between (1) and (2) in Theorem 1 will be presented in Section 6, and in Section 7, we shall prove Theorem 3.

2. PROPERTIES OF THE EXAMPLES

The purpose of this section is to prove the two examples constructed in the first section.

2.1. The proof of Example 1.1. It follows from [31, Theorem 5.11] that f is an isometry with respect to the quasihyperbolic metric. Hence the first statement holds.

For the proof of (2), we let $a = \frac{1}{t}$, $x = 1$ and $b = t$, where $t > 1$. Suppose on the contrary that f is weakly H -quasisymmetric for some $H > 0$. Since $|a - x| \leq |b - x|$, we have

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq H,$$

which implies that for any $t > 1$,

$$t \leq H.$$

This is impossible. Hence the proof is complete. \square

2.2. The proof of Example 1.2. Obviously, the first statement (a) follows from [31, Example 7.4].

Now, we prove (c). Suppose on the contrary that f is q -locally weakly H -quasisymmetric in G for some $H \geq 1$ and $0 < q < 1$. Let $0 < \varepsilon < \frac{1}{2}$, $O = (n, \frac{1}{2})$ and $\mathbb{B}(O, \varepsilon) = \{z \in \mathbb{C} : |z - O| < \varepsilon\}$. Then $\mathbb{B}(O, \varepsilon) \subset G$, and so

$$q\mathbb{B}(O, \varepsilon) = \mathbb{B}(O, q\varepsilon) \subset G.$$

Let

$$a = (n, \frac{1}{2} + \frac{1}{2}q\varepsilon) \text{ and } b = (n - \frac{1}{2}q\varepsilon, \frac{1}{2}).$$

Then we easily know that

$$a, O, b \in q\mathbb{B}(O, \varepsilon) \text{ and } |a - O| = |b - O|.$$

But elementary computations show that

$$\frac{|f(a) - f(O)|}{|f(b) - f(O)|} = \frac{2\sqrt{5}}{5}(n+1),$$

which implies that f is not weakly H -quasisymmetric in $q\mathbb{B}(O, \varepsilon)$ for any $H \geq 1$.

Next, we show (b). Suppose that f is ψ -semisolid in every proper subdomain of G for some homeomorphism $\psi : [0, \infty) \rightarrow [0, \infty)$. Then (a) and Theorem 1 or [31, Theorem 7.9] guarantee that f is q -locally weakly H -quasisymmetric in G for some $H \geq 1$ and $0 < q < 1$. This contradicts (c). Hence the proof of the example is complete. \square

3. AUXILIARY RESULTS

3.1. Basic geometric results in metric spaces. We always denote the open metric ball with center $x \in X$ and radius $r > 0$ by

$$\mathbb{B}(x, r) = \{z \in X : |z - x| < r\}$$

and for $\lambda > 0$,

$$\lambda\mathbb{B}(x, r) = \{z \in X : |z - x| < \lambda r\}.$$

For a set A in X , we always use ∂A (resp. \overline{A}) to denote the boundary (resp. the closure) of A .

By a curve, we mean any continuous function $\gamma : [a, b] \rightarrow X$. The length of γ is denoted by

$$\ell(\gamma) = \sup \left\{ \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| \right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < t_2 \dots < t_n = b$. The curve is *rectifiable* if $\ell(\gamma) < \infty$. In particular, if the metric is taken to be the quasihyperbolic metric, the length of γ is denoted by $\ell_{qh}(\gamma)$.

The length function associated with a rectifiable curve $\gamma : [a, b] \rightarrow X$ is $s_\gamma : [a, b] \rightarrow [0, \ell(\gamma)]$, given by $s_\gamma(t) = \ell(\gamma|_{[a, t]})$. For any rectifiable curve $\gamma : [a, b] \rightarrow X$, there is a unique curve $\gamma_s : [0, \ell(\gamma)] \rightarrow X$ such that $\gamma = \gamma_s \circ s_\gamma$. Obviously, $\ell(\gamma_s|_{[0, t]}) = t$ for $t \in [0, \ell(\gamma)]$. The curve γ_s is called the arclength parametrization of γ .

For a rectifiable curve γ in X , the line integral over γ of each Borel function $\varrho : X \rightarrow [0, \infty)$ is

$$\int_\gamma \varrho ds = \int_0^{\ell(\gamma)} \varrho \circ \gamma_s(t) dt.$$

Let X be a connected metric space and Ω a nonempty proper subset of X . It follows from [14, Remark 2.2] that if Ω is open, then $\partial\Omega \neq \emptyset$. Further, we have

Lemma 3.1. *Suppose X is a connected metric space. For any nonempty set $\Omega \subsetneq X$, $\partial\Omega \neq \emptyset$.*

Proof. Suppose on the contrary that $\partial\Omega = \emptyset$. Let $u \in \Omega$. Then there is an open set $V \subset X$ such that $u \in V$ and either $V \subset \Omega$ or $V \subset X \setminus \Omega$. Since $u \in V \cap \Omega$, we see that $V \subset \Omega$. This implies that Ω is open. By [14, Remark 2.2], it is impossible. Hence the lemma holds. \square

For $z \in \Omega$, we always use $\delta_\Omega(z)$ to denote the distance from z to $\partial\Omega$. We make the convention that $\delta_\Omega(z) = 0$ if and only if $z \in \partial\Omega$. Then we easily have the following useful inequality: For any $x, y \in \Omega$,

$$\delta_\Omega(x) \leq \delta_\Omega(y) + |x - y|.$$

For $z \in \Omega$, assume that $\delta_\Omega(z) > 0$. Let $r \in (0, \delta_\Omega(z))$. Then the metric ball $\mathbb{B}(z, r)$ is not necessarily contained in Ω , and also the intersection $\mathbb{B}(z, r) \cap \Omega$ is not always connected even when Ω is a domain. So we always consider the component of the intersection $\mathbb{B}(z, r) \cap \Omega$ containing the center z , which is denoted by $\mathbb{B}^\Omega(z, r)$,

when Ω is open. Similarly, we use $\overline{\mathbb{B}}^\Omega(z, r)$ (resp. $\partial\mathbb{B}^\Omega(z, r)$) to denote the closure (resp. the boundary) of the component of $\mathbb{B}(z, r) \cap \Omega$ containing the center z . In particular, for $\lambda > 0$, we always use $\lambda\mathbb{B}^\Omega(z, r)$ to denote the component $\mathbb{B}^\Omega(z, \lambda r)$ provided $\lambda r < \delta_\Omega(z)$. Then we have the following.

Lemma 3.2. *Suppose X is a c -quasiconvex metric space and $\Omega \subsetneq X$ is open. For any rectifiably connected set $D \subset \mathbb{B}(z, r)$ with $z \in D \cap \Omega$, if $r \leq \delta_\Omega(z)$, then*

$$D \subset \mathbb{B}^\Omega(z, r) \subset \Omega.$$

Proof. Obviously, it suffices to prove that $D \subset \Omega$. We show this by contradiction. Suppose on the contrary that D is not contained in Ω . Then there exists a point $u \in D \subset \mathbb{B}(z, r)$, but $u \notin \Omega$. Let γ be a rectifiable curve in D joining the points z and u . Since $\partial\Omega \neq \emptyset$ and $u \notin \Omega$, we see that there must exist a point u_0 such that $u_0 \in \gamma \cap \partial\Omega$. Hence

$$|z - u_0| \geq \delta_\Omega(z) \geq r > |z - u_0|,$$

which is the desired contradiction. \square

For two domains D and G in a space, in general, the assumption $D \subset G$ implies $\mathbb{B}^D(z, \delta_D(z)) \subset \mathbb{B}^\Omega(z, \delta_\Omega(z))$. This basic and very useful property holds for many types of spaces, for example in Banach spaces. But this property is not valid in metric spaces as the following example shows.

Example 3.1. *Let*

$$X = l_1 \cup l_2 \cup l_3 \cup l_4, \quad \Omega = X \setminus l_5 \quad \text{and} \quad D = l_1,$$

where $l_1 = \{(x, 0) : -2 < x < 2\}$, $l_2 = \{(x, 1) : -2 < x < 2\}$, $l_3 = \{(-2, y) : 0 \leq y \leq 1\}$, $l_4 = \{(2, y) : 0 \leq y \leq 1\}$ and $l_5 = \{(x, 1) : -1 \leq x \leq 1\}$. Then, obviously, we have

- (1) X is 5-quasiconvex with the Euclidean metric;
- (2) $D \subsetneq \Omega$, $\delta_D(z) = 2$ and $\delta_\Omega(z) = \sqrt{2}$, where $z = (0, 0)$, and
- (3) $\mathbb{B}^\Omega(z, \delta_\Omega(z)) \subsetneq \mathbb{B}^D(z, \delta_D(z))$.

Our next lemma illustrates that with some constraint, this phenomenon can be avoided.

Lemma 3.3. *Suppose X is a c -quasiconvex metric space and $\Omega \subsetneq X$ is open.*

- (1) *For any nonempty subset P in Ω , we have $\delta_P(z) \leq c\delta_\Omega(z)$ for any $z \in P$;*
- (2) *For any open subset D in Ω , we have $\mathbb{B}^D\left(z, \frac{r}{c}\delta_D(z)\right) \subset \mathbb{B}^\Omega(z, r\delta_\Omega(z))$ for any $z \in D$, where $0 < r \leq 1$.*

Proof. Obviously, the second statement easily follows from the first one by taking $P = D$. So, to prove this lemma, it suffices to show the first statement. Without loss of generality, we may assume that $P \subsetneq \Omega$ and $\delta_P(z) > 0$. Let $a \in \partial P$. Then we claim that

$$(3.1) \quad \delta_P(z) \leq c|z - a|.$$

It follows from Lemma 3.1 that $\partial P \neq \emptyset$. To prove this inequality, we consider two cases. For the first case where $a \in \partial P$, obviously,

$$\delta_P(z) \leq |z - a|.$$

For the remaining case, that is, $a \notin \partial P$, we know that $a \in X \setminus \overline{P}$. Let α denote a curve in X connecting a and z such that

$$\ell(\alpha) \leq c|z - a|.$$

Since $z \in P$, obviously, $\alpha \cap \partial P \neq \emptyset$. Assume that $b \in \alpha \cap \partial P$. Then

$$\delta_P(z) \leq |z - b| < \ell(\alpha) \leq c|z - a|.$$

Hence the inequality (3.1) holds.

By taking the infimum over all $a \in \partial\Omega$ in (3.1), we easily know that the statement (1) in the lemma is true, and so the proof of the lemma is complete. \square

Further, we have the following result.

Lemma 3.4. *Suppose X is a c -quasiconvex metric space and $\Omega \subsetneq X$ is open.*

- (1) *Suppose $z \in \Omega$ and $0 < r \leq \frac{2}{2+c}\delta_\Omega(z)$. Then $\mathbb{B}(z, r) \subset \Omega$;*
- (2) *Suppose $z \in \Omega$ and $0 < r \leq \frac{1}{1+c}\delta_\Omega(z)$. Then for any x and $y \in \mathbb{B}(z, r)$, there must exist a curve $\gamma \subset \mathbb{B}^\Omega(z, (1+c)r)$ such that*

$$\ell(\gamma) \leq c|x - y|;$$

- (3) *Suppose that $z \in \Omega$ and $0 < r \leq \frac{1}{1+c}\delta_\Omega(z)$. Then*

$$\mathbb{B}(z, r) \subset \mathbb{B}^\Omega(z, (1+c)r).$$

In particular, for any $z \in \Omega$ and $0 < r \leq \frac{1}{1+c}\delta_\Omega(z)$,

$$\mathbb{B}(z, r) \subset \mathbb{B}^\Omega(z, \delta_\Omega(z)).$$

Proof. We prove the first statement by contradiction. Suppose there exists some $x \in \mathbb{B}(z, r)$ such that $x \notin \Omega$. By the assumption, we see that there is a curve $\beta \subset X$ connecting z and x such that

$$\ell(\beta) \leq c|x - z|.$$

Then $\ell(\beta) < cr$. We claim that for $w \in \beta$,

$$(3.2) \quad |w - z| < \frac{2+c}{2}r.$$

To show this claim, let w_0 be the point in β such that

$$\ell(\beta_{[z, w_0]}) = \ell(\beta_{[w_0, x]}),$$

where $\beta_{[z, w_0]}$ denotes the part of β with the endpoints z and w_0 . According to the position of w in β , we consider two possibilities. For the first possibility where $w \in \beta_{[z, w_0]}$, we easily see that

$$|w - z| \leq \frac{1}{2}\ell(\beta) < \frac{c}{2}r.$$

For the other possibility, that is, $w \in \beta_{[x, w_0]}$, we can easily get that

$$|w - z| \leq |w - x| + |x - z| < \frac{1}{2}\ell(\beta) + r \leq \frac{2+c}{2}r.$$

Hence (3.2) holds, and the claim is proved.

Now, we proceed with the proof based on (3.2). Since $z \in \Omega$ and $\partial\Omega \neq \emptyset$, we know that $\beta \cap \partial\Omega \neq \emptyset$. Assume that $z_0 \in \beta \cap \partial\Omega$. Then we deduce from (3.2) that

$$|z_0 - z| \geq \delta_\Omega(z) \geq \frac{2+c}{2}r > |z_0 - z|.$$

This is the desired contradiction, which shows that the statement (1) in the lemma is true.

To get proofs of the second and the third statements in the lemma, for any x and $y \in \mathbb{B}(z, r)$, we let γ denote a curve in X joining x to y such that

$$\ell(\gamma) \leq c|x - y|.$$

Then $\ell(\gamma) < 2cr$. Now, we check that

$$(3.3) \quad \gamma \subset \mathbb{B}(z, (1+c)r) \cap \Omega.$$

Since for all $w \in \gamma$,

$$(3.4) \quad |w - z| \leq \min\{|w - x|, |w - y|\} + r \leq \frac{1}{2}\ell(\gamma) + r < (1+c)r,$$

which implies that $\gamma \subset \mathbb{B}(z, (1+c)r)$.

It remains to prove the inclusion $\gamma \subset \Omega$. Again, we show this by contradiction. Suppose that γ is not contained in Ω . Since it follows from the statement (1) that x and $y \in \Omega$, there must be a point u_0 which is contained in the intersection $\gamma \cap \partial\Omega$, and so (3.4) implies

$$\delta_\Omega(z) \leq |u_0 - z| < (1+c)r \leq \delta_\Omega(z).$$

This is the desired contradiction. Hence $\gamma \subset \mathbb{B}(z, (1+c)r) \cap \Omega$.

For any point $v \in \gamma$, let ζ denote a curve in X connecting z and v such that

$$\ell(\zeta) \leq c|v - z|.$$

By replacing x (resp. y) by z (resp. v) in (3.3), we can know that

$$\zeta \subset \mathbb{B}(z, (1+c)r) \cap \Omega.$$

The arbitrariness of $v \in \gamma$ makes sure that

$$\gamma \subset \mathbb{B}^\Omega(z, (1+c)r).$$

So the second statement in the lemma holds.

Obviously, the statement (3) follows from (3.3) and the arbitrariness of the points x and y in $\mathbb{B}(z, r)$, and hence the proof of our lemma is complete. \square

In the following, we get several applications of Lemma 3.4, which will be useful in the discussions later on.

Lemma 3.5. *Suppose X is a c -quasiconvex metric space and $\Omega \subsetneq X$ is open. For any metric ball $\mathbb{B}(z, r)$ with $z \in \Omega$ and $0 < r \leq \delta_\Omega(z)$, let $B = \mathbb{B}^\Omega(z, r)$. Then*

- (1) $\delta_B(z) = r$;
- (2) *for any $u \in \overline{B}$, $|u - z| = r$ if and only if $u \in \partial B$.*

Proof. By Lemma 3.1, we know that $\partial B \neq \emptyset$. Obviously, we only need to prove (1) since (2) is a direct consequence of (1). We prove (1) by contradiction.

Suppose $\delta_B(z) < r$. Then there must exist $z_0 \in \partial B$ such that

$$|z_0 - z| \leq \delta_B(z) + \frac{r - \delta_B(z)}{2} < r.$$

Since $z_0 \in \partial B$ and $B \subset \Omega$, we know that $z_0 \in \overline{\Omega}$. Obviously, $z_0 \notin \partial\Omega$, because otherwise, $\delta_\Omega(z) \leq |z - z_0| < r$. Hence $z_0 \in \Omega$.

Let $r_1 = r - |z_0 - z|$. Then $r_1 > 0$ and

$$\delta_\Omega(z_0) > \delta_\Omega(z) - |z - z_0| \geq r_1,$$

and so it follows that

$$\mathbb{B}(z_0, r_1) \subset \mathbb{B}(z, r).$$

Further, let $r_2 = \frac{1}{1+c}r_1$. Then Lemma 3.4 (3) implies

$$\mathbb{B}(z_0, r_2) \subset \mathbb{B}^\Omega(z_0, r_1).$$

Since $z_0 \in \partial B$, we see that there is a sequence $\{x_i\}_{i=1}^\infty \subset B$ such that $\lim_{i \rightarrow \infty} x_i = z_0$. Then for all sufficiently large i , $x_i \in \mathbb{B}(z_0, r_2)$, which implies $\mathbb{B}^\Omega(z_0, r_1) \cap B \neq \emptyset$. Hence it follows from the fact $\mathbb{B}(z_0, r_1) \subset \mathbb{B}(z, r)$ that $\mathbb{B}^\Omega(z_0, r_1) \subset B$, and so

$$\mathbb{B}(z_0, r_2) \subset B.$$

This is impossible since $z_0 \in \partial B$. Hence (1) holds, and so the proof of lemma is finished. \square

Lemma 3.6. *Suppose X is a c -quasiconvex metric space and $\Omega \subsetneq X$ is open. Then for any $0 < \mu \leq 1$, $\mathbb{B}^\Omega(z, \mu\delta_\Omega(z))$ is a domain in Ω for any $z \in \Omega$.*

Proof. Obviously, it suffices to show the openness of $\mathbb{B}^\Omega(z, \mu\delta_\Omega(z))$. Since for any $w \in B = \mathbb{B}^\Omega(z, \mu\delta_\Omega(z))$, $w \notin \partial B$. Otherwise, Lemma 3.5 guarantees that

$$\mu\delta_\Omega(z) = \delta_B(z) \leq |z - w| < \mu\delta_\Omega(z).$$

Hence this lemma is proved. \square

By Lemma 3.6, we have the following.

Lemma 3.7. *Suppose X is a c -quasiconvex metric space. Then X must be locally connected.*

Proof. Let z be any point in X , and let U be any open set in X with $z \in U$. To prove this lemma, we only need to find a domain W in U with $z \in W$.

Since U is open and X is c -quasiconvex, it follows from [18, page 120] that there is $r > 0$ such that $\mathbb{B}(z, r) \subsetneq U$. Let

$$V = \mathbb{B}(z, r) \text{ and } W = \mathbb{B}^V(z, \mu\delta_V(z)),$$

where $\mu\delta_V(z) < r$. Then we deduce from Lemma 3.6 that this W is our needed. Hence Lemma 3.7 is true. \square

3.2. Quasihyperbolic metric. The *quasihyperbolic length* of a rectifiable curve or a path γ in the metric in a domain $G \subsetneq X$ is the number:

$$\ell_{k_G}(\gamma) = \int_{\gamma} \frac{|dz|}{\delta_G(z)}.$$

For any z_1, z_2 in G , the *quasihyperbolic distance* $k_G(z_1, z_2)$ between z_1 and z_2 is defined by

$$k_G(z_1, z_2) = \inf\{\ell_{k_G}(\gamma)\},$$

where the infimum is taken over all rectifiable curves γ joining z_1 to z_2 in G .

Gehring and Palka [8] introduced the quasihyperbolic metric of a domain in \mathbb{R}^n . For the basic properties of this metric we refer to [7]. Recall that a curve γ from z_1 to z_2 is a *quasihyperbolic geodesic* if $\ell_{k_G}(\gamma) = k_G(z_1, z_2)$. Each subcurve of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between any two points in E exists if the dimension of X is finite, see [7, Lemma 1]. This is not true in arbitrary metric spaces (cf. [30, Example 2.9]).

We establish comparison results between the metrics $|\cdot|$ and k_G in a c -quasiconvex metric space. It is a modified version of [14, Theorems 2.7 and 2.8].

Lemma 3.8. *Let X be a c -quasiconvex metric space and let $G \subsetneq X$ be a domain.*

(1) *For all $x, y \in G$,*

$$(3.5) \quad |x - y| \leq (e^{k_G(x, y)} - 1)\delta_G(x);$$

(2) *Suppose $z \in G$ and $0 < t < 1$. Then for $x, y \in \overline{\mathbb{B}}^G(z, \frac{t}{2c}\delta_G(z))$,*

$$(3.6) \quad \frac{c}{c+t} \cdot \frac{|x-y|}{\delta_G(z)} \leq k_G(x, y) \leq \frac{c}{1-\frac{1+c}{2c}t} \cdot \frac{|x-y|}{\delta_G(z)};$$

(3) *Suppose that $x, y \in G$ and either $|x-y| \leq \frac{1}{3c}\delta_G(x)$ or $k_G(x, y) \leq 1$. Then*

$$(3.7) \quad \frac{1}{2} \frac{|x-y|}{\delta_G(x)} < k_G(x, y) \leq 3c \frac{|x-y|}{\delta_G(x)}.$$

Proof. The inequality (3.5) follows from [14, Theorem 2.7]. In the following, we prove (3.6) and (3.7).

Since X is a c -quasiconvex metric space, we see from Lemma 3.4 (2) that there is a curve γ in $\mathbb{B}^G(z, \frac{1+c}{2c}t\delta_G(z))$ joining x and y with

$$\ell(\gamma) \leq c|x-y|.$$

Since for $w \in \gamma$,

$$\delta_G(w) \geq \delta_G(z) - |w - z| \geq \left(1 - \frac{1+c}{2c}t\right)\delta_G(z),$$

we see that

$$k_G(x, y) \leq \int_{\gamma} \frac{|dw|}{\delta_G(w)} \leq \frac{c}{1 - \frac{1+c}{2c}t} \cdot \frac{|x - y|}{\delta_G(z)}.$$

This shows that the right-side inequality of (3.6) is true. Now, we give a proof of the left-side inequality of (3.6). It follows from [14, Observation 2.6] that G is rectifiably connected. Hence it suffices to prove that for any rectifiable curve $\alpha \subset G$ joining x and y ,

$$\ell_{k_G}(\alpha) \geq \frac{c}{c+t} \cdot \frac{|x - y|}{\delta_G(z)}.$$

To this end, let α be a rectifiable curve in G joining x and y . We divide the proof into two cases. For the first case where $\alpha \subset \overline{\mathbb{B}}^G(z, \frac{t}{c}\delta_G(z))$, we see that

$$(3.8) \quad \delta_G(w) \leq |w - z| + \delta_G(z) \leq \left(1 + \frac{t}{c}\right)\delta_G(z)$$

for all $w \in \alpha$, and hence

$$\ell_{k_G}(\alpha) = \int_{\alpha} \frac{|dw|}{\delta_G(w)} \geq \frac{c}{c+t} \cdot \frac{|x - y|}{\delta_G(z)},$$

as required.

For the remaining case, that is, $\alpha \not\subset \overline{\mathbb{B}}^G(z, \frac{t}{c}\delta_G(z))$, obviously, α has two sub-curves α_1 and α_2 in $\overline{\mathbb{B}}^G(z, \frac{t}{c}\delta_G(z))$ joining the sets $\partial\mathbb{B}^G(z, \frac{t}{2c}\delta_G(z))$ and $\partial\mathbb{B}^G(z, \frac{t}{c}\delta_G(z))$. Again, (3.8) implies

$$\ell_{k_G}(\alpha) \geq \int_{\alpha_1 \cup \alpha_2} \frac{|dw|}{\delta_G(w)} \geq \frac{c}{c+t} \cdot \frac{|x - y|}{\delta_G(z)},$$

since $\ell(\alpha_i) \geq \frac{t}{2c}\delta_G(z) \geq \frac{|x-y|}{2}$ for $i = 1, 2$. Hence the proof of (3.6) is complete.

To prove (3.7), we first consider the case where $|x - y| \leq \frac{1}{3c}\delta_G(x)$. By taking $t = \frac{2}{3}$ in (3.6), obviously, we have that

$$\frac{|x - y|}{2\delta_G(x)} < \frac{c|x - y|}{(c + \frac{2}{3})\delta_G(x)} \leq k_G(x, y) \leq \frac{3c^2}{2c - 1} \cdot \frac{|x - y|}{\delta_G(x)} \leq 3c \frac{|x - y|}{\delta_G(x)}.$$

For the remaining case, that is, $|x - y| > \frac{1}{3c}\delta_G(x)$ and $k_G(x, y) \leq 1$, the right-side inequality of (3.7) is obvious. Since for $r \in (0, 1]$,

$$e^r - 1 < 2r,$$

it follows from (3.5) that

$$|x - y| \leq (e^{k_G(x, y)} - 1)\delta_G(x) < 2k_G(x, y)\delta_G(x),$$

which implies the left-side inequality of (3.7) is true too. Hence the proof of (3.7) is complete. \square

The following result is on the quasiconvexity of a domain G in X with respect to the quasihyperbolic metric k_G .

Lemma 3.9. *Let X be a c -quasiconvex metric space and let $G \subsetneq X$ be a domain.*

- (1) *Let γ be a rectifiable path in G . Then $\ell_{k_G}(\gamma)$ is the length of γ in the metric space (G, k_G) , i.e., $\ell_{k_G}(\gamma) = \ell_{qh}(\gamma)$;*
- (2) *Then the space (G, k_G) is λ -quasiconvex for all $\lambda > 1$.*

Proof. The proof of the statement (1) easily follows from (3.6) in Lemma 3.8 and [4, Lemma 2.6], and the second statement easily follows from the first one. \square

We remark that in Lemma 3.9 and in what follows, the notation (G, e) is applied to emphasize the related metric e .

3.3. Length metric. In the following, we always use d to denote the length metric in X with respect to $|\cdot|$ and k'_G to denote the quasihyperbolic metric with respect to d in G , where $G \subsetneq X$ is a domain. Then we have the following analog of statements (i) and (ii) in [13, Lemma 9.2] in the setting of c -quasiconvex metric spaces, i.e. without the assumption that X is proper.

Lemma 3.10. *Suppose X is a c -quasiconvex metric space and $G \subsetneq X$ is a domain. Then*

- (1) *for all $x, y \in X$, $|x - y| \leq d(x, y) \leq c|x - y|$; and*
- (2) *for all $x, y \in G$, $\frac{1}{c}k_G(x, y) \leq k'_G(x, y) \leq ck_G(x, y)$.*

Proof. To prove this lemma, it suffices to verify the inequality (2) in the lemma since the proof of (1) is obvious.

By the inequality (1) in the lemma, it follows that for any curve $\gamma \subset X$,

$$\ell(\gamma) \leq \ell_d(\gamma) \leq c\ell(\gamma),$$

where $\ell_d(\gamma)$ denotes the arclength of γ with respect to d . Let $\delta'_G(x)$ denote the distance from x to the boundary of G with respect to d . Then we have that for all $x \in G$,

$$(3.9) \quad \delta_G(x) \leq \delta'_G(x) \leq c\delta_G(x).$$

For the proof, we let ε be a positive number. Then there are points $a, b \in \partial G$ such that

$$|x - a| \leq \delta_G(x) + \frac{\varepsilon}{c} \quad \text{and} \quad d(x, b) \leq \delta'_G(x) + \varepsilon,$$

respectively.

By (1), we easily know that

$$\delta_G(x) \leq |x - b| \leq d(x, b) \leq \delta'_G(x) + \varepsilon$$

and

$$\delta'_G(x) \leq d(x, a) \leq c|x - a| \leq c\delta_G(x) + \varepsilon.$$

Thus the inequality (3.9) easily follows from the arbitrariness of ε .

Now, we can finish the proof by using (3.9). Let $|d_d(z)|$ denote the arclength differential with respect to the length metric d . Then

$$|dz| \leq |d_d(z)| \leq c|dz|.$$

Since for all $x, y \in G$ and for any $h > 0$, there is a curve γ (resp. γ') connecting x and y in G such that

$$\ell_{k_G}(\gamma) \leq k_G(x, y) + \frac{h}{c} \quad (\text{resp. } \ell_{k'_G}(\gamma') \leq k'_G(x, y) + \frac{h}{c}),$$

we know that

$$k'_G(x, y) \leq \int_{\gamma} \frac{|d_d(z)|}{\delta'_G(z)} \leq \int_{\gamma} \frac{c|dz|}{\delta_G(z)} = c\ell_{k_G}(\gamma) \leq ck_G(x, y) + h$$

and

$$k_G(x, y) \leq \int_{\gamma'} \frac{|dz|}{\delta_G(z)} \leq \int_{\gamma'} \frac{c|d_d(z)|}{\delta'_G(z)} = c\ell_{k'_G}(\gamma') \leq ck'_G(x, y) + h.$$

Obviously, letting $h \rightarrow 0$ yields the inequality (2) in the lemma, and so the proof of the lemma is complete. \square

4. THE PROOF OF THE IMPLICATION FROM (1) TO (3) IN THEOREM 1

This section consists of two subsections. In the first subsection, we shall show the implication from relativity and locally weak quasisymmetry with a constraint on the coefficient of quasiconvexity. In the second subsection, the proof of the implication from (1) to (3) in Theorem 1 will be proved.

4.1. Relativity and locally weak quasisymmetry. We start this subsection with the definition of relativity of homeomorphisms.

Definition 4.1. Suppose $f : G \rightarrow G'$ is a homeomorphism, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains. Then f is said to be

- (1) $(\theta; t_0)$ -relative if there are $t_0 \in (0, 1]$ and a homeomorphism $\theta : [0, t_0] \rightarrow [0, \infty)$ such that

$$\frac{|f(x) - f(y)|}{\delta_{G'}(f(x))} \leq \theta\left(\frac{|x - y|}{\delta_G(x)}\right)$$

whenever $x, y \in G$ with $|x - y| < t_0\delta_G(x)$;

- (2) θ -relative if $t_0 = 1$.

The following is the main result in this subsection.

Lemma 4.1. Suppose that X is a c -quasiconvex metric space with $1 \leq c \leq \frac{1+\sqrt{3}}{2}$ and that both $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If $f : G \rightarrow G'$ is θ -relative in every subdomain of G , then f is q' -locally η' -quasisymmetric in G with $q' = q'(c) = \frac{1}{(2+c)^3}$ and $\eta' = \eta'_{\theta, c}$.

Proof. Since $\theta : [0, 1) \rightarrow [0, \infty)$ is homeomorphic, without loss of generality, we assume that

$$M = \theta\left(\frac{2c}{1+2c}\right) > 1.$$

For distinct points x, a, b in $\mathbb{B}^G(z, q\delta_G(z))$ with $|x - a| = t|x - b|$, where $z \in G$ and $t > 0$, to prove that f is locally quasimetric in G , we need a relationship between $|f(a) - f(x)|$ and $|f(x) - f(b)|$. To this end, we divide the discussions into three cases according to the location of the parameter t . We first discuss the case where $0 < t \leq \frac{2c}{1+2c}$. In this case, we have

Claim 4.1. *Suppose $0 < t \leq \frac{2c}{1+2c}$ and $0 < q_1 \leq \frac{1}{2+c}$. Then*

- (1) *x and a must lie in a component of $G \setminus \{b\}$, denoted by G_{abx} . Further, G_{abx} is open;*
- (2) $|f(a) - f(x)| \leq \theta(t)|f(x) - f(b)|$.

We prove (1) in the claim by contradiction. Suppose on the contrary that x and a lie in two different components of $G \setminus \{b\}$. By Lemma 3.4 (2), there exists a curve $\alpha \subset G$ joining x and a with

$$\ell(\alpha) \leq c|x - a|.$$

Then from the contrary assumption we know that $b \in \alpha$. Moreover,

$$|x - b| < \ell(\alpha) \leq c|x - a| = ct|x - b|,$$

and so

$$\frac{1}{c} < t \leq \frac{2c}{1+2c}.$$

This is the desired contradiction since we have assumed that $c \leq \frac{\sqrt{3}+1}{2}$. Hence we know that x and a lie in the same component of $G \setminus \{b\}$, which is denoted by G_{abx} .

Still, we need to show that G_{abx} is open. This directly follows from Lemma 3.7 and [18, Theorem 25.4]. Hence G_{abx} is a domain in $G \setminus \{b\}$, and so the first statement in the claim is true.

It follows from the first statement in the lemma that G_{abx} is a subdomain of $G \setminus \{b\}$. Then the assumptions in the lemma guarantee that the restriction $f|_{G_{abx}}$ is θ -relative. For the moment, we need an auxiliary result.

Assertion 4.1. $b \in \partial G_{abx} \subset \partial(G \setminus \{b\}) = \partial G \cup \{b\}$.

Obviously, the relation $b \in \partial G_{abx}$ follows from the rectifiable connectedness of G (cf. [14, Observation 2.6]).

For convenience, let $G \setminus \{b\} = G_b$. In the following, we prove the inclusion $\partial G_{abx} \subset \partial(G_b)$ by contradiction. Suppose on the contrary that there is a point $w \in (\partial G_{abx}) \cap G_b$. It follows that $r = \delta_{G_b}(w) > 0$, and thus there is a point $u \in \mathbb{B}(w, \frac{1}{1+c}r)$, but $u \notin G_{abx}$.

Since $w \in \partial G_{abx}$, we see that there are $w_n \in G_{abx}$ such that $w_n \rightarrow w$ as $n \rightarrow \infty$. Without loss of generality, we may assume that all $w_n \in \mathbb{B}(w, \frac{1}{1+c}r)$. Since $u, w_n \in \mathbb{B}(w, \frac{1}{1+c}r)$, we see from Lemma 3.4 (2) that there is a curve $\tau'_n \subset \mathbb{B}^{G_b}(w, r) \subset G_b$ connecting w_n and u .

On the other hand, the first statement in the claim and [14, Observation 2.6] guarantee that G_{abx} is rectifiably connected. Since a and $w_n \in G_{abx}$ (resp. x and $w_n \in G_{abx}$), we know that there is a curve τ_n'' (resp. τ_n''') in G_{abx} connecting a and w_n (resp. x and w_n). Then $\tau_n'' \cup \tau_n' \subset G_b$ (resp. $\tau_n''' \cup \tau_n' \subset G_b$) connects a and u (resp. x and u). Hence $u \in G_{abx}$. This contradiction finishes the proof of the assertion.

We continue the proof of the claim with the aid of Assertion 4.1. Since

$$\delta_G(x) \geq \delta_G(z) - |x - z| \geq \delta_G(z) - q\delta_G(z) > 2q_1\delta_G(z) > |x - b|,$$

we infer from Assertion 4.1 that

$$|x - a| = t|x - b| = t\delta_{G_{abx}}(x) < \delta_{G_{abx}}(x),$$

and, again, this assertion guarantees that

$$(4.1) \quad |f(a) - f(x)| \leq \theta(t)\delta_{f(G_{abx})}(f(x)) \leq \theta(t)|f(x) - f(b)|.$$

Hence the proof of the claim is complete.

Obviously, $\theta(t) \leq M$.

Second, we consider the case where $\frac{2c}{1+2c} < t \leq 1$. In this case, we have

Claim 4.2. *If $\frac{2c}{1+2c} < t \leq 1$ and $0 < q_2 \leq \frac{1}{(2+c)^2}$, then*

$$|f(a) - f(x)| \leq H|f(x) - f(b)|,$$

where $H = H(\theta, c) \geq M$.

In this case, by constructing a curve connecting x and a and choosing suitable points from this curve, we shall apply the similar reasoning as in Claim 4.1 to any two successive points to obtain our desired inequality.

By Lemma 3.4 (2), there is a curve $\beta \subset \mathbb{B}^G(z, (1+c)q_2\delta_G(z))$ joining x and a with

$$\ell(\beta) \leq c|x - a|.$$

Define inductively the successive points $x = x_0, \dots, x_k = a$ of β so that x_j is the last point of β in $\overline{\mathbb{B}}(x_{j-1}, (\frac{2c}{1+2c})^j|x - b|)$. Obviously, $k \geq 2$, and for $1 \leq j \leq k-1$,

$$|x_{j-1} - x_j| = \left(\frac{2c}{1+2c}\right)^j |x - b|$$

and

$$|x_{k-1} - x_k| \leq \left(\frac{2c}{1+2c}\right)^k |x - b|.$$

Now, we start to get an upper bound for k . Since for $1 \leq j \leq k-1$,

$$l(\beta[x_{j-1}, x_j]) \geq |x_{j-1} - x_j| = \left(\frac{2c}{1+2c}\right)^j |x - b|,$$

we have

$$\sum_{j=1}^{k-1} \left(\frac{2c}{1+2c}\right)^j |x - b| \leq l(\beta) \leq c|x - a| = ct|x - b| < c|x - b|.$$

Hence we obtain

$$(4.2) \quad k < \frac{\log 2}{\log(1+2c) - \log(2c)} + 1 = k_0.$$

For points $x_1, x_0 = x$ and b , since $|x_1 - x_0| = \frac{2c}{1+2c}|x - b|$ and $(1+c)q_2 < \frac{1}{2+c}$, the similar method in the proof of (4.1) shows that

$$(4.3) \quad |f(x_1) - f(x_0)| \leq M|f(x) - f(b)|,$$

and for points x_0, x_1 and x_2 , since $|x_2 - x_1| \leq \frac{2c}{1+2c}|x_1 - x_0|$, similarly, we have

$$(4.4) \quad |f(x_2) - f(x_1)| \leq M|f(x_1) - f(x_0)| \leq M^2|f(x) - f(b)|.$$

Now, we claim that for $1 \leq j \leq k$,

$$(4.5) \quad |f(x_j) - f(x_{j-1})| \leq M^j|f(x) - f(b)|.$$

By the relations (4.3) and (4.4), we see that to check (4.5), it suffices to discuss the case where $k > 2$. In this case, for $1 \leq j \leq k-1$, as in Claim 4.1, we use $G_j = G_{x_{j-1}x_jx_{j+1}}$ to denote the component of $G \setminus \{x_j\}$ containing the points x_{j-1} and x_{j+1} . Then by Lemma 3.7 and [18, Theorem 25.4], we know that for each j , G_j is a domain in $G \setminus \{x_j\}$, and thus the assumptions in the lemma imply that the restriction $f|_{G_j}$ is θ -relative. For points x_{j+1}, x_j and x_{j-1} , since

$$|x_j - x_{j+1}| \leq \frac{2c}{1+2c}|x_j - x_{j-1}|,$$

it follows from

$$\begin{aligned} \delta_G(x_j) - |x_j - x_{j-1}| &\geq \delta_G(x_0) - \sum_{i=0}^{j-1} |x_i - x_{i+1}| - |x_j - x_{j-1}| \\ &\geq \delta_G(z) - |x_0 - z| - \sum_{i=0}^{j-1} |x_i - x_{i+1}| - |x_j - x_{j-1}| \\ &> (1 - (1+4c)q_2)\delta_G(z) \geq 0 \end{aligned}$$

together with Assertion 4.1 that for $j \geq 2$,

$$|x_j - x_{j+1}| \leq \frac{2c}{1+2c}|x_j - x_{j-1}| = \frac{2c}{1+2c}\delta_{D_{j-1}}(x_j) < \delta_{D_{j-1}}(x_j),$$

which shows that

$$\begin{aligned} |f(x_3) - f(x_2)| &\leq M\delta_{f(D_1)}(f(x_2)) \leq M|f(x_2) - f(x_1)| \leq M^3|f(x) - f(b)|, \\ &\dots \\ |f(x_k) - f(x_{k-1})| &\leq M\delta_{f(D_{k-2})}(f(x_{k-1})) \leq M^k|f(x) - f(b)|. \end{aligned}$$

Hence the relation (4.5) is true.

It follows from (4.5) that

$$|f(a) - f(x)| \leq \sum_{j=1}^k |f(x_j) - f(x_{j-1})| < kM^k|f(x) - f(b)|.$$

Combining the estimate (4.2) on k , we have

$$(4.6) \quad |f(x) - f(a)| \leq H|f(x) - f(b)|,$$

where $H = k_0 M^{k_0} \geq M$. The Claim 4.2 is proved.

By Claims 4.1 and 4.2, in fact, we have proved the following conclusion.

Corollary 4.1. *The restriction $f|_{\mathbb{B}^G(z, \frac{1}{(2+c)^2}\delta_G(z))}$ is weakly H -quasisymmetric with $H = k_0 M^{k_0} \geq M$.*

Finally, we consider the case where $t > 1$.

Claim 4.3. *If $t > 1$, then for any $x, a, b \in \mathbb{B}^G(z, \frac{1}{(2+c)^3}\delta_G(z))$,*

$$|f(a) - f(x)| \leq (1 + ct)H^{1+ct}|f(b) - f(x)|.$$

We shall use Corollary 4.1 to prove this claim. For any $x, a, b \in \mathbb{B}^G(z, \frac{1}{(2+c)^3}\delta_G(z))$, Lemma 3.4 (2) implies that there is a curve $\gamma \subset \mathbb{B}^G(z, \frac{1}{(2+c)^2}\delta_G(z))$ joining x and a with

$$\ell(\gamma) \leq c|x - a|.$$

Define inductively the successive points $x = u_0, \dots, u_\varrho = a$ of γ so that each u_i denotes the last point of γ in $\overline{\mathbb{B}}(u_{i-1}, |b - x|)$. Then

$$\varrho \geq 2, \quad |u_{i-1} - u_i| = |b - x| \quad \text{for } 1 \leq i \leq \varrho - 1 \quad \text{and} \quad |u_{\varrho-1} - u_\varrho| \leq |b - x|.$$

Now, we come to obtain an upper bound for ϱ . Since for $1 \leq i \leq \varrho - 1$,

$$\ell(\gamma[u_{i-1}, u_i]) \geq |u_{i-1} - u_i| = |b - x|,$$

we have

$$(\varrho - 1)|b - x| \leq \ell(\gamma) \leq c|x - a| = ct|b - x|,$$

which implies that

$$(4.7) \quad \varrho \leq 1 + ct.$$

Since b and all u_i ($i \in \{0, \dots, \varrho\}$) are contained in $\mathbb{B}^G(z, \frac{1}{(2+c)^2}\delta_G(z))$, Corollary 4.1 guarantees that

$$|f(a) - f(x)| \leq \sum_{i=1}^{\varrho} |f(u_i) - f(u_{i-1})| \leq \varrho H^\varrho |f(x) - f(b)|.$$

Thus, (4.7) implies

$$(4.8) \quad |f(a) - f(x)| \leq (1 + ct)H^{(1+ct)}|f(b) - f(x)|.$$

The proof of Claim 4.3 is complete.

Now, we are able to finish the proof. Let

$$\eta'(t) = \begin{cases} \frac{H}{M}\theta(t), & \text{if } 0 < t \leq \frac{2c}{1+2c}, \\ k_0 t + k_1, & \text{if } \frac{2c}{1+2c} \leq t \leq 1, \\ (1 + ct)H^{(1+ct)}, & \text{if } t > 1, \end{cases}$$

where

$$k_0 = (1 + 2c)[(1 + c)H^{1+c} - H] \text{ and } k_1 = -2c(1 + c)H^{1+c} + (1 + 2c)H.$$

Then it follows from (4.1), (4.6) and (4.8) that for any $x, a, b \in \mathbb{B}^G(z, \frac{1}{(2+c)^3}\delta_G(z))$ with $|a - x| = t|b - x|$,

$$|f(a) - f(x)| \leq \eta'(t)|f(b) - f(x)|.$$

This shows that Lemma 4.1 holds. \square

4.2. The proof of the implication from (1) to (3) in Theorem 1. By using the length metric, we shall show Lemma 4.2 below, from which the proof of the implication from (1) to (3) in Theorem 1 easily follows. We start with the definition of “strong geodesic condition” (cf. [6]).

Definition 4.2. Suppose X is a rectifiably connected metric space and $G \subsetneq X$ is a domain. We say that G satisfies the strong geodesic condition if every ball $\mathbb{B}(x_0, r)$ of G satisfies the following condition: for every $x \in \mathbb{B}(x_0, r)$ there exists a curve γ in $\mathbb{B}(x_0, r)$ connecting x and x_0 such that $\ell(\gamma) < r$.

The following result is related to the property of the strong geodesic condition for curves.

Claim 4.4. Suppose that X is a c -quasiconvex metric space and that both $D \subsetneq X$ and $D' \subsetneq Y$ are domains. If

- (1) for any $x_0 \in D$ and $0 < r < \delta_D(x_0)$, $\mathbb{B}(x_0, r) \subset D$;
- (2) D satisfies the strong geodesic condition; and
- (3) $f : D \rightarrow D'$ is φ -semisolid,

then f is θ -relative with $\theta = \theta_{\varphi, c}$.

For the proof, we let $x, y \in D$ with $|x - y| = t\delta_D(x)$, where $0 < t < 1$. We shall find a homeomorphism $\theta : [0, 1) \rightarrow [0, \infty)$ such that

$$\frac{|f(x) - f(y)|}{\delta_{D'}(f(x))} \leq \theta\left(\frac{|x - y|}{\delta_D(x)}\right) = \theta(t),$$

where $\theta = \theta_{\varphi, c}$.

To reach this goal, first, we estimate the quasihyperbolic distance between x and y , which is as follows:

$$(4.9) \quad k_D(x, y) \leq \frac{1+t}{1-t}.$$

Since $y \in \mathbb{B}(x, \frac{1+t}{2}\delta_D(x)) \subset D$, we see from the assumptions in the claim that there is a curve γ connecting x and y in $\mathbb{B}(x, \frac{1+t}{2}\delta_D(x))$ such that

$$\ell(\gamma) < \frac{1+t}{2}\delta_D(x).$$

Hence

$$\delta_D(z) \geq \delta_D(x) - |x - z| \geq \frac{1-t}{2}\delta_D(x)$$

for all $z \in \gamma$, and as a consequence, we have

$$k_D(x, y) \leq \int_{\gamma} \frac{|dz|}{\delta_D(z)} \leq \frac{1+t}{1-t},$$

as required.

Second, by (3.7) in Lemma 3.8, we have that if $0 \leq t \leq \frac{1}{3c}$, then

$$(4.10) \quad k_D(x, y) \leq 3ct.$$

Now, we are ready to construct the needed homeomorphism. Let $\psi(t) = e^t - 1$ and

$$\theta_0(t) = \begin{cases} \frac{3c(3c+1)}{3c-1}t, & \text{if } 0 \leq t \leq \frac{1}{3c}, \\ \frac{1+t}{1-t}, & \text{if } \frac{1}{3c} < t < 1. \end{cases}$$

Then by (3.5) in Lemma 3.8, we know that

$$\frac{|f(x) - f(y)|}{\delta_{D'}(f(x))} \leq \psi(k_{D'}(f(x), f(y))) \leq \psi \circ \varphi \circ \theta_0(t),$$

and obviously, $\theta = \psi \circ \varphi \circ \theta_0$ is what we want. Hence Claim 4.4 is complete.

Next, we are going to show that, in the setting of length metric, the assumptions (1) and (2) in Claim 4.4 are naturally satisfied.

Claim 4.5. *Suppose that X is a c -quasiconvex metric space and that $D \subsetneq X$ is a domain. Then*

- (1) *for any $x_0 \in D$ and $0 < r < \delta'_D(x_0)$, $\mathbb{B}_d(x_0, r) = \{x \in X : d(x, x_0) < r\} \subset D$; and*
- (2) *D satisfies the strong geodesic condition with respect to d .*

Here and in what follows, we always use the notation $\text{id}_{|\cdot|}$ to denote the identity map which is from $(Q, |\cdot|)$ to (Q, d) for any domain Q in X .

Since (X, d) is λ -quasiconvex for any $\lambda > 1$, Lemma 3.10 (1) implies that the map

$$\text{id}_{|\cdot|} : (D, |\cdot|) \rightarrow (D, d)$$

is λ -bilipschitz, which shows that (D, d) is also a domain.

For any $x_0 \in D$, clearly, the ball $\mathbb{B}_d(x_0, r)$ is rectifiably connected, since every two points in $\mathbb{B}_d(x_0, r)$ can be joined by a curve through x_0 with length at most $2r$. Then Lemma 3.2 makes sure that for all $r \in (0, \delta'_D(x_0))$, the ball $\mathbb{B}_d(x_0, r)$ is contained in D . Hence (1) holds.

To prove (2) in the claim, we let y be an arbitrary point in $\mathbb{B}_d(x_0, r)$. Since $d(x_0, y) < r$, we see that there is a curve γ in X joining y and x_0 such that

$$\ell_d(\gamma) \leq d(y, x_0) + \frac{1}{2}(r - d(y, x_0)) < r.$$

which implies $\gamma \subset \mathbb{B}_d(x_0, r)$, and so, we see that D satisfies the strong geodesic condition with respect to d , and hence the proof of Claim 4.5 is complete.

Suppose that X is a c -quasiconvex metric space and that both $G \subsetneq X$ and $G' \subsetneq Y$ are domains. For a homeomorphism $f : G \rightarrow G'$, let

$$g = f \circ \text{id}_{|\cdot|}^{-1} : (G, d) \rightarrow (G', |\cdot|).$$

Then we have the following claim.

Claim 4.6. *If f is φ -semisolid in every subdomain D of G , then g is θ' -relative in D , where $\theta' = \theta'_{\varphi, c}$.*

We are going to apply Claims 4.4 and 4.5 to show this claim. We assume that D is an arbitrary subdomain of G . Since for all $x, y \in (D, d)$,

$$k_{f(D)}(g(x), g(y)) \leq \varphi(k_D(\text{id}_{|\cdot|}^{-1}(x), \text{id}_{|\cdot|}^{-1}(y))) \leq \varphi(ck'_D(x, y)) = \varphi'(k'_D(x, y)),$$

where $\varphi'(t) = \varphi(ct)$. This implies that g is φ' -semisolid in D . Then it follows from Claim 4.5 that g satisfies all assumptions in Claim 4.4 if the metric $|\cdot|$ is replaced by the one d . Claim 4.4 tells us that g is θ' -relative in D , where $\theta' = \theta'_{\varphi, c}$. Hence the claim is proved.

Lemma 4.2. *Suppose that X is a c -quasiconvex metric space and that both $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If $f : G \rightarrow G'$ is φ -semisolid in every subdomain of G , then*

- (1) *there is a homeomorphism $\eta = \eta_{\varphi, c} : [0, \infty) \rightarrow [0, \infty)$ such that f is q -locally η -quasisymmetric in G , where $q = \frac{1}{(2+c_0)^3 c}$;*
- (2) *there is a constant $H = H(\varphi, c)$ such that f is q -locally weakly H -quasisymmetric in G .*

Proof. We first prove (1) in the lemma. Obviously, (X, d) is c_0 -quasiconvex with $c_0 = \frac{\sqrt{3}+1}{2}$. By applying Claim 4.6, we see that the induced map

$$g = f \circ \text{id}_{|\cdot|}^{-1} : (G, d) \rightarrow (G', |\cdot|)$$

is θ -relative in every subdomain of G with $\theta = \theta_{\varphi, c_0} = \theta_{\varphi}$. So Lemma 4.1 is available now. It follows from this lemma that g is q' -locally η -quasisymmetric with $q' = \frac{1}{(2+c_0)^3}$ and $\eta' = \eta'_{c_0, \theta} = \eta'_{\varphi}$.

Now, we show that f is q -locally η -quasisymmetric, where $q = \frac{1}{(2+c_0)^3 c}$ and $\eta(t) = \eta'(ct)$.

For any $x, a, b \in \mathbb{B}^G(z, q\delta_G(z))$ with $|x - a| \leq t|x - b|$, it follows from Lemma 3.10 (1) and (3.9) that

$$d(w, z) \leq c|w - z| < cq\delta_G(z) \leq q'\delta'_G(z)$$

for all $w \in \mathbb{B}^G(z, q\delta_G(z))$, which implies that

$$\mathbb{B}^G(z, q\delta_G(z)) \subset \mathbb{B}_d(z, q'\delta'_G(z))$$

as a set. Again using Lemma 3.10 (1) we get

$$d(x, a) \leq c|x - a| \leq ct|x - b| \leq ctd(x, b),$$

and then

$$|f(x) - f(a)| = |g \circ \text{id}_{|\cdot|}(x) - g \circ \text{id}_{|\cdot|}(a)| \leq \eta'(ct)|g \circ \text{id}_{|\cdot|}(x) - g \circ \text{id}_{|\cdot|}(b)| = \eta(t)|f(x) - f(b)|,$$

where $\eta(t) = \eta'(ct)$. Hence (1) in the lemma is proved.

The proof of the second statement in this lemma easily follows from (1) by taking $H = \max\{\eta(1), 1\}$. Hence the proof of Lemma 4.2 is complete. \square

5. THE PROOF OF THE IMPLICATION FROM (3) TO (1) IN THEOREM 1

This section consists of four subsections. In the first subsection, a mapping property of locally weakly quasisymmetric mappings is given, and the proof of the implication from (3) to (1) in Theorem 1 is contained in the rest three subsections.

5.1. A mapping property of locally weakly quasisymmetric mappings.

Lemma 5.1. *Suppose that X is a c -quasiconvex and complete metric space, and that $f : G \rightarrow G'$ q -locally weakly H -quasisymmetric with $H \geq 1$ and $0 < q < 1$, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains. For $z \in G$, let $Q = \mathbb{B}^G(z, r)$, where $0 < r \leq \frac{q}{1+2c}\delta_G(z)$. Then $f(\partial Q) = \partial f(Q)$ and $\partial f(\overline{Q}) \subset f(\overline{Q})$.*

Proof. Let $\{f(w_j)\}_{j=1}^\infty$ be a Cauchy sequence in $f(\overline{Q})$ and $\{w_j\}_{j=1}^\infty$ the corresponding sequence in \overline{Q} . Then we have

Claim 5.1. *The sequence $\{w_j\}_{j=1}^\infty$ is Cauchy.*

We prove this claim by contradiction. Suppose $\{w_j\}_{j=1}^\infty$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ such that for each positive integer k , there is $j(k) > k$ satisfying

$$|w_k - w_{j(k)}| \geq \varepsilon.$$

Obviously, there is $z_k \in \{w_k, w_{j(k)}\}$ such that

$$(5.1) \quad |z_k - w_1| \geq \varepsilon/2.$$

Let $t = \frac{2r}{\varepsilon}$. Then we have

$$|w_1 - w_{j(1)}| \leq 2r = t\varepsilon \leq 2t|z_k - w_1|.$$

For their images under f , we have the following estimate.

Assertion 5.1. $|f(w_1) - f(w_{j(1)})| \leq (1 + 2ct)H^{1+2ct}|f(z_k) - f(w_1)|$.

We divide the proof into two cases. For the first case where $2t \leq 1$, since $|w_1 - w_{j(1)}| \leq 2t|z_k - w_1| \leq |z_k - w_1|$ and since $w_1, w_{j(1)}$ and $z_k \in \overline{\mathbb{B}}^G(z, \frac{q}{1+2c}\delta_G(z)) \subset \mathbb{B}^G(z, q\delta_G(z))$, we have

$$|f(w_1) - f(w_{j(1)})| \leq H|f(z_k) - f(w_1)|,$$

as required.

For the remaining case, that is, $2t > 1$, by Lemma 3.4 (2), there is a curve γ in $\mathbb{B}^G(z, \frac{1+c}{1+2c}q\delta_G(z))$ joining w_1 and $w_{j(1)}$ such that

$$\ell(\gamma) \leq c|w_1 - w_{j(1)}|.$$

Define inductively the successive points $w_1 = a_0, \dots, a_s = w_{j(1)}$ of γ so that for each $i \in \{1, \dots, s\}$, a_i denotes the last point of γ in $\overline{\mathbb{B}}(a_{i-1}, |z_k - w_1|)$. Obviously,

$s \geq 2$. The following upper bound of s is needed in the proof later on. Since for $i \in \{1, \dots, s-1\}$,

$$|a_i - a_{i-1}| = |z_k - w_1| \leq \ell(\gamma_{[a_{i-1}, a_i]}),$$

we see that

$$(s-1)|z_k - w_1| \leq \ell(\gamma) \leq c|w_1 - w_{j(1)}| \leq 2ct|z_k - w_1|.$$

Hence

$$(5.2) \quad s \leq 1 + 2ct.$$

Since all z_k and a_i ($i \in \{0, \dots, s\}$) are contained in $\mathbb{B}^G(z, \frac{1+c}{1+2c}q\delta_G(z)) \subset \mathbb{B}^G(z, q\delta_G(z))$, we know that

$$\begin{aligned} |f(w_1) - f(w_{j(1)})| &\leq \sum_{i=1}^s |f(a_i) - f(a_{i-1})| \leq sH^s |f(z_k) - f(w_1)| \\ &\leq (1 + 2ct)H^{1+2ct} |f(z_k) - f(w_1)|. \end{aligned}$$

Hence the assertion holds.

Let us proceed with the proof of Claim 5.1. Since

$$|z_k - w_1| \leq 2r = t\varepsilon \leq t|w_k - w_{j(k)}|,$$

by replacing $2t$ with t , the similar reasoning as in the proof of Assertion 5.1 shows that

$$|f(z_k) - f(w_1)| \leq (1 + ct)H^{1+ct} |f(w_k) - f(w_{j(k)})|.$$

Since $\{f(w_j)\}_{j=1}^\infty$ is a Cauchy sequence, we see that $|f(w_k) - f(w_{j(k)})| \rightarrow 0$ as $k \rightarrow \infty$, and so $f(z_k) \rightarrow f(w_1)$. Then it follows from Assertion 5.1 that $f(w_1) = f(w_{j(1)})$, which is a contradiction since f is homeomorphic. The claim is proved.

Since X is complete, Claim 5.1 implies that $\{w_j\}_{j=1}^\infty$ converges to a point $u \in \overline{Q}$, and so $f(w_j) \rightarrow f(u) \in f(\overline{Q})$, which shows that the lemma is true. \square

5.2. Locally weak quasisymmetry and ring property. We start this subsection with the definition of ring property.

Definition 5.1. Suppose $f : G \rightarrow G'$ is a homeomorphism, where $G \subsetneq X$ and $G' \subsetneq Y$ are domains. f is said to have the $(M; \alpha; \beta)$ -ring property in G if there are constants $1 < \alpha \leq \beta$ and $M > 0$ such that for any $B = \mathbb{B}^G(z, r)$ with $z \in G$ and $\beta r < \delta_G(z)$,

$$\text{diam}(f(\overline{B})) \leq M \text{dist}(f(\overline{B}), \partial f(\alpha B)),$$

where $\text{diam}(U)$ (resp. $\text{dist}(U, V)$) means the diameter (resp. the distance) of the set U (resp. between the two disjoint sets U and V).

Lemma 5.2. Under the assumptions in Theorem 1, and further, if f is q -locally weakly H -quasisymmetric in G with $H \geq 1$ and $0 < q < 1$, then f has the $(M; \alpha; \beta)$ -ring property in every subdomain of G , where all the constants M , α and β depend only on c , q and H .

Proof. The assumptions in the lemma imply that

$$(5.3) \quad |f(x) - f(a)| \leq H|f(x) - f(b)|$$

for any triplet x, a and b of points in $\mathbb{B}^G(z, q\delta_G(z))$ with $|x - a| \leq |x - b|$ and for any $z \in G$. We shall show that for any subdomain D of G , f has the $(M; \alpha; \beta)$ -ring property in D with

$$M = 2H^2(H + 1), \quad \alpha = 3 \quad \text{and} \quad \beta = 6cq^{-1}.$$

For $w \in D$, let $K = \mathbb{B}^D(w, r)$, where $0 < r \leq \frac{q}{6(1+c)}\delta_D(w)$. Then

$$\overline{K} \subset \overline{3K} \subset \mathbb{B}^D(w, \frac{q}{2c}\delta_D(w)).$$

Since Lemma 3.3 shows $\mathbb{B}^D(w, \frac{q}{2c}\delta_D(w)) \subset \mathbb{B}^G(w, q\delta_G(w))$, we know that

$$\overline{K} \subset \overline{3K} \subset \mathbb{B}^G(w, q\delta_G(w)).$$

We first show a claim.

Claim 5.2. $\text{diam}(f(\overline{K})) \leq 2H^2(H + 1) \text{dist}(f(\overline{K}), f(\partial(3K)))$.

Obviously, it suffices to show that for all $a, b, y \in \overline{K}$ and $z \in \partial(3K)$,

$$|f(a) - f(b)| \leq 2H^2(H + 1)|f(y) - f(z)|.$$

Since for any $u \in \partial\overline{K}$, it follows from Lemma 3.5 that

$$\max\{|a - w|, |b - w|\} \leq |u - w| \leq |z - u|.$$

Then we know from (5.3) that

$$|f(a) - f(b)| \leq |f(a) - f(w)| + |f(b) - f(w)| \leq 2H|f(u) - f(w)| \leq 2H^2|f(z) - f(u)|.$$

Meanwhile, by Lemma 3.5, we have $|y - u| \leq |z - y|$, which implies

$$|f(y) - f(u)| \leq H|f(y) - f(z)|,$$

whence

$$|f(z) - f(u)| \leq |f(z) - f(y)| + |f(y) - f(u)| \leq (H + 1)|f(y) - f(z)|.$$

Hence

$$|f(a) - f(b)| \leq 2H^2(H + 1)|f(y) - f(z)|,$$

as required, and so the claim holds.

We are ready to finish the proof of Lemma 5.2. By Lemma 5.1 and Claim 5.2, we see that

$$\text{diam}(f(\overline{K})) \leq 2H^2(H + 1) \text{dist}(f(\overline{K}), f(\partial(3K))) = 2H^2(H + 1) \text{dist}(f(\overline{K}), \partial f(3K)).$$

Hence the proof of the lemma is complete. \square

5.3. Ring property and relativity.

Lemma 5.3. *Under the assumptions in Theorem 1, and further, if f is q -locally weakly H -quasimetric with $H \geq 1$ and $0 < q < 1$, then f is $(\theta; t_0)$ -relative in every subdomain of G , where $\theta = \theta_{c,c',M,\alpha,\beta}$ and $t_0 = t_0(c, c', M, \alpha, \beta)$.*

Proof. By Lemma 5.2, we may assume that f has the $(M; \alpha; \beta)$ -ring property in every subdomain of G , where $M = 2H^2(H + 1)$, $\alpha = 3$ and $\beta = 6cq^{-1}$. Let D be a subdomain of G and

$$t_0 = \frac{1}{2c(2c\alpha)^3\beta}.$$

To prove this lemma, it suffices to prove that f is (θ, t_0) -relative in D for some homeomorphism $\theta: [0, t_0) \rightarrow [0, \infty)$, where $\theta = \theta_{c,c',M,\alpha,\beta}$. To find such a homeomorphism, we let x and $y \in D$ with $|x - y| = t\delta_D(x)$, where $t \in (0, t_0)$, and let m be the largest integer with

$$2c(2c\alpha)^m\beta t < 1.$$

Obviously, $m \geq 3$. Set

$$B_0 = \mathbb{B}^D(x, t\delta_D(x))$$

and for $1 \leq j \leq m$,

$$s_j = (2c\alpha)^j.$$

Then

$$2cs_j\beta t\delta_D(x) \leq 2cs_m\beta t\delta_D(x) < \delta_D(x).$$

Let $z'_0 \in \partial f(D)$ be such that

$$|z'_0 - f(x)| \leq 2\delta_{f(D)}(f(x)).$$

Then there is a curve $\gamma \subset Y$ joining $f(x)$ and z'_0 such that

$$\ell(\gamma) \leq c'|z'_0 - f(x)| \leq 2c'\delta_{f(D)}(f(x)).$$

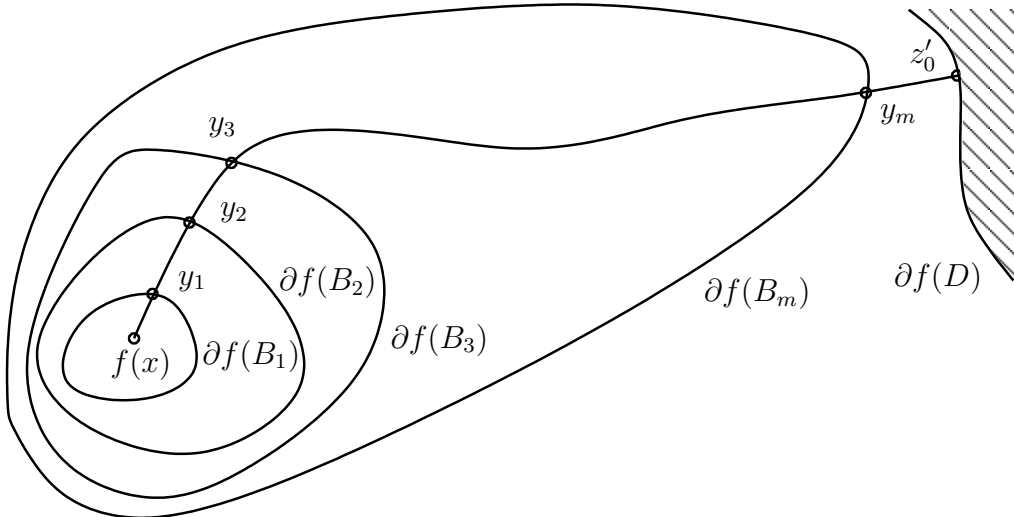


FIGURE 1. The curve γ in Y and the related points.

Let

$$B_j = \mathbb{B}^D(x, s_j t \delta_D(x)).$$

Then it follows that for each $j \leq m$,

$$(5.4) \quad \text{diam}(f(\overline{B}_j)) \leq M \text{dist}(f(\overline{B}_j), \partial f(\alpha B_j)) \leq M \text{dist}(f(\overline{B}_j), \partial f(2c\alpha B_j)),$$

and so

$$\begin{aligned} \text{dist}(f(\overline{B}_j), \partial f(D)) &\geq \text{dist}(f(\overline{B}_m), \partial f(D)) \geq \text{dist}(f(\overline{B}_m), \partial f(2c\alpha B_m)) \\ &\geq \frac{1}{M} \text{diam}(f(\overline{B}_m)) > 0, \end{aligned}$$

which implies that for each $j \in \{1, 2, \dots, m\}$, the intersection set $\gamma \cap \partial f(B_j)$ is not empty. Let $y_j \in \gamma \cap \partial f(B_j)$ (see Figure 1). Then it follows from (5.4) that for each $3 \leq j \leq m$,

$$\begin{aligned} |y_2 - f(x)| &\leq \text{diam}(f(\overline{B}_{j-1})) \leq M \text{dist}(f(\overline{B}_{j-1}), \partial f(2c\alpha B_{j-1})) \\ &= M \text{dist}(f(\overline{B}_{j-1}), \partial f(B_j)). \end{aligned}$$

Since for each $j \leq m$, $s_j t < \frac{q}{12c^2}$, we see that $\overline{B}_j \subset \mathbb{B}^D\left(x, \frac{q}{1+2c} \delta_D(x)\right)$, and then Lemma 5.1 implies that

$$\partial f(\overline{B}_{j-1}) \subset f(\overline{B}_{j-1}).$$

Hence

$$|y_2 - f(x)| \leq M \text{dist}(\partial f(\overline{B}_{j-1}), \partial f(B_j)) \leq M \text{dist}(\partial f(B_{j-1}), \partial f(B_j)) \leq M |y_j - y_{j-1}|.$$

Summing over the indices j , we obtain that

$$(m-2)|y_2 - f(x)| \leq M \sum_{j=3}^m |y_j - y_{j-1}| < M \ell(\gamma) \leq 2M c' \delta_{f(D)}(f(x)).$$

Moreover, since $y \in \overline{\mathbb{B}}(x, t \delta_D(x))$, Lemma 3.4 (3) guarantees that $y \in B_1$, and so it follows that

$$|f(x) - f(y)| \leq \text{diam}(f(\overline{B}_1)) \leq M \text{dist}(f(\overline{B}_1), \partial f(B_2)) \leq M |y_2 - f(x)|.$$

Hence

$$\frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))} \leq \frac{2M^2}{m-2} c'.$$

Since $2c(2c\alpha)^3 \beta t < 1 \leq 2c(2c\alpha)^{m+1} \beta t$, we have

$$m-2 \geq \frac{\log(1/(2c\beta t)) - 3 \log(2c\alpha)}{\log(2c\alpha)},$$

which implies that

$$\frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))} \leq \frac{2M^2 c' \log(2c\alpha)}{\log(1/(2c(2c\alpha)^3 \beta t))}.$$

Now, let

$$\theta(t) = \frac{2M^2 c' \log(2c\alpha)}{\log(1/(2c(2c\alpha)^3 \beta t))}.$$

Obviously, this θ is the desired. Hence Lemma 5.3 holds. \square

5.4. Relativity and semisolidity.

Lemma 5.4. *Under the assumptions in Theorem 1, and further, if f is $(\theta; t_0)$ -relative in every subdomain of G , then there exists a homeomorphism $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in every subdomain of G , where $\varphi = \varphi_{c', \theta, t_0}$.*

Proof. Assume that f is $(\theta; t_0)$ -relative in every subdomain of G . Then it suffices to prove that f is φ -semisolid in any subdomain D of G , where $\varphi = \varphi_{c', t_0, \theta}$. To this end, we choose

$$t_1 = \frac{1}{2} \min \left\{ t_0, \theta^{-1} \left(\frac{1}{3c'} \right) \right\},$$

and let x and y be any two points in D with $k_D(x, y) < t_1$. Then it follows from Lemma 3.8 (3.7) that

$$\frac{|x - y|}{\delta_D(x)} \leq 2k_D(x, y) < 2t_1 \leq t_0,$$

which implies

$$|f(x) - f(y)| \leq \theta(2k_D(x, y))\delta_{f(D)}(f(x)) \leq \theta(2t_1)\delta_{f(D)}(f(x)) \leq \frac{1}{3c'}\delta_{f(D)}(f(x)).$$

Again, Lemma 3.8 (3.7) leads to

$$k_{f(D)}(f(x), f(y)) \leq 3c' \frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))} \leq 3c'\theta(2k_D(x, y)).$$

Hence, f is (ψ, t_1) -uniformly continuous in the QH metric with $\psi(t) = 3c'\theta(2t)$ (See, for example, [25] or [26] for the definition). Obviously, $\psi(0) = 0$. Since (D, k_D) is λ -quasiconvex for any $\lambda > 1$, we see from [25, Lemma 2.5] or [26, Lemma 3.2] that there is a homeomorphism $\varphi = \varphi_{t_1, \psi} : [0, \infty) \rightarrow [0, \infty)$ such that f is φ -semisolid in D . Obviously, $\varphi = \varphi_{c', t_0, \theta}$, and thus Lemma 5.4 is proved. \square

6. THE PROOF OF THE EQUIVALENCE BETWEEN (1) AND (2) IN THEOREM 1

Since the the implication from (2) to (1) is obvious, we only need to show the implication from (1) to (2). To prove this implication, it suffices to find constants $K \geq 1$ and $0 < \alpha \leq 1$ depending only on φ, c, c' such that for any subdomain D of G ,

$$(6.1) \quad k_{f(D)}(f(x), f(y)) \leq K \max\{k_D(x, y)^\alpha, k_D(x, y)\}$$

for all $x, y \in D$. To reach this goal, we need some preparation.

Assume that D is a subdomain of G . Then it follows from the first statement in Lemma 4.2 that f is q -locally η -quasisymmetric with $q = \frac{1}{(2+c_0)^3c}$ and $\eta = \eta_{\theta, c}$, where $c_0 = \frac{1+\sqrt{3}}{2}$.

Let

$$(6.2) \quad B_x = \mathbb{B}^D \left(x, \frac{q}{(2+c)^2} \delta_D(x) \right) \quad \text{and} \quad f_1 = f|_{B_x}.$$

Then [22, Corollary 3.12] guarantees that there exist constants $K_0 \geq 1$ and $0 < \alpha \leq 1$ depending only on η such that f_1 is η_0 -quasisymmetric, where

$$\eta_0(t) = K_0 \max\{t^\alpha, t^{1/\alpha}\}.$$

Next, we show that f_1 can be extended to $\overline{B_x}$, and this extension is also η_0 -quasisymmetric. Since X is complete, so is $\overline{B_x}$. Using the facts that the inverse of a quasisymmetric mapping is also quasisymmetric and the image of a Cauchy sequence under a quasisymmetric mapping is also Cauchy, we know that $\overline{f(B_x)}$ is complete. Hence [31, Theorem 6.12] makes sure that f_1 has a natural extension to $\overline{B_x}$, still denoted by f_1 , which is also η_0 -quasisymmetric.

We leave the proof of the inequality (6.1) for a moment and obtain an estimate on the quantity $\frac{|f(x)-f(y)|}{\delta_{f(D)}(f(x))}$.

Let

$$t_0 = \min \left\{ \varphi^{-1} \left(\frac{1}{3c'} \right), \frac{q}{(2+c)^4} \right\}.$$

Obviously, $t_0 \leq \frac{1}{81}$. Then for x and y in D with

$$|x - y| = t\delta_D(x),$$

we have

Claim 6.1. *If $t \leq 2t_0$, then*

$$\frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))} \leq c' K_0 \left((2+c)^2 q^{-1} \frac{|x-y|}{\delta_D(x)} \right)^\alpha.$$

For the proof, we fix $\varepsilon > 0$ and then choose a point $u' \in \partial f(D)$ with

$$|f(x) - u'| \leq \delta_{f(D)}(f(x)) + \varepsilon.$$

Since Y is c' -quasiconvex, we see that there exists a curve γ in Y joining $f(x)$ and u' with

$$\ell(\gamma) \leq c'|f(x) - u'|.$$

Obviously, $\gamma \cap \partial f(B_x) \neq \emptyset$ since $u' \notin f(B_x)$. Furthermore, it follows from Lemma 5.1 that $\partial f(B_x) = f(\partial B_x)$, and so $\gamma \cap f(\partial B_x) \neq \emptyset$. Now, we assume that $v' \in \gamma \cap f(\partial B_x)$. Then there is a point v in ∂B_x such that $v' = f(v)$. Hence

$$|f(x) - f(v)| \leq \ell(\gamma) \leq c'|f(x) - u'| \leq c'(\delta_{f(D)}(f(x)) + \varepsilon).$$

On the other hand, it follows from Lemma 3.4 (3) that

$$\overline{\mathbb{B}}\left(x, \frac{2q}{(2+c)^4} \delta_D(x)\right) \subset \mathbb{B}\left(x, \frac{q}{(2+c)^3} \delta_D(x)\right) \subset B_x,$$

which implies $y \in B_x$. Further, Lemma 3.5 indicates that

$$|x - v| = \frac{q}{(2+c)^2} \delta_D(x).$$

Hence we know from the fact f_1 being η_0 -quasisymmetric in $\overline{B_x}$ that

$$\frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x)) + \varepsilon} \leq c' \frac{|f(x) - f(y)|}{|f(x) - f(v)|} \leq c' \eta_0 \left(\frac{|x-y|}{|x-v|} \right) \leq c' \eta_0 \left((2+c)^2 q^{-1} \frac{|x-y|}{\delta_D(x)} \right),$$

from which the claim follows since $(2 + c)^2 q^{-1} \frac{|x-y|}{\delta_D(x)} < 1$.

Now, we continue the proof of the inequality (6.1) by dividing the discussions into two cases.

Case 6.1. $k_D(x, y) \leq t_0$.

By the choice of t_0 and the φ -semisolidity of f in D , we have

$$k_{f(D)}(f(x), f(y)) \leq \varphi(k_D(x, y)) \leq \varphi(t_0) \leq \frac{1}{3c'},$$

and then the inequality (3.7) in Lemma 3.8 gives

$$(6.3) \quad k_{f(D)}(f(x), f(y)) \leq 3c' \frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))}.$$

Since $k_D(x, y) \leq t_0 < 1$, again, we know from Lemma 3.8 (3.7) that

$$\frac{|x - y|}{\delta_D(x)} \leq 2k_D(x, y) \leq 2t_0.$$

Applying Claim 6.1, it follows from (6.3) that

$$(6.4) \quad k_{f(D)}(f(x), f(y)) \leq 3c'^2 K_0 \left((2 + c)^2 q^{-1} \frac{|x - y|}{\delta_D(x)} \right)^\alpha < K_1 k_D(x, y)^\alpha,$$

where $K_1 = 3c'^2 K_0 (2 + c)^{2\alpha} q^{-\alpha}$.

Case 6.2. $k_D(x, y) > t_0$.

By Lemma 3.9, we see that (D, k_D) is λ -quasiconvex for all $\lambda > 1$. Obviously, the φ -semisolidity of f in D implies that for any $u, v \in D$, if $k_D(u, v) \leq t_0$, then $k_{f(D)}(f(u), f(v)) \leq \varphi(t_0)$. By taking $q = t_0$ and $C = \varphi(t_0)$ in [31, Lemma 2.3], we know that

$$k_{f(D)}(f(x), f(y)) \leq \lambda \frac{\varphi(t_0)}{t_0} k_D(x, y) + \varphi(t_0) \leq (\lambda + 1) \frac{\varphi(t_0)}{t_0} k_D(x, y).$$

Letting $\lambda \rightarrow 1$ yields

$$(6.5) \quad k_{f(D)}(f(x), f(y)) \leq K_2 k_D(x, y),$$

where $K_2 = \frac{2\varphi(t_0)}{t_0}$.

Now, we are able to complete the proof of the theorem. By letting

$$K = \max\{K_1, K_2\},$$

we see that (6.1) easily follows from (6.4) and (6.5), and hence the equivalence between (1) and (2) in Theorem 1 is proved. \square

7. THE PROOF OF THEOREM 3 AND AN APPLICATION

In this section, we prove Theorem 3. We begin with the following result showing the invariance of semisolidity under the composition operator of mappings.

Lemma 7.1. *Suppose X_i are c_i -quasiconvex metric spaces, and $G_i \subsetneq X_i$ are domains, where $i = 1, 2, 3$. If $f : G_1 \rightarrow G_2$ is φ_1 -semisolid in every subdomain of G_1 and $g : G_2 \rightarrow G_3$ is φ_2 -semisolid in every subdomain of G_2 . Then the composition $g \circ f : G_1 \rightarrow G_3$ is $\varphi_2 \circ \varphi_1$ -semisolid in every subdomain of G_1 .*

Proof. For any subdomain D in G_1 , by the hypotheses, we see that $f : D \rightarrow f(D) \subsetneq G_2$ is φ_1 -semisolid and $g : f(D) \rightarrow g \circ f(D) \subset G_3$ is φ_2 -semisolid, which shows that for all $x, y \in D$,

$$k_{f(D)}(f(x), f(y)) \leq \varphi_1(k_D(x, y))$$

and

$$k_{g \circ f(D)}(g \circ f(x), g \circ f(y)) \leq \varphi_2(k_{f(D)}(f(x), f(y))).$$

Hence

$$k_{g \circ f(D)}(g \circ f(x), g \circ f(y)) \leq \varphi_2 \circ \varphi_1(k_D(x, y)),$$

and so the proof of Lemma 7.1 is complete since the domain D is arbitrarily taken from G_1 . \square

The proof of Theorem 3: Under the assumptions, by Theorem 1, there are homeomorphisms φ_1 and φ_2 from $[0, \infty)$ to $[0, \infty)$ such that f is φ_1 -semisolid in every subdomain of G_1 and g is φ_2 -semisolid in every subdomain of G_2 , where φ_1 and φ_2 depend at most on $c_1, c_2, c_3, q_1, q_2, H_1$ and H_2 . Lemma 7.1 yields that the composition $g \circ f$ is $\varphi_2 \circ \varphi_1$ -semisolid in every subdomain of G_1 . Again, it follows from Theorem 1 that there is a constant $0 < q < 1$ such that $g \circ f$ is q -locally η -quasisymmetric in G_1 , where $\eta : [0, \infty) \rightarrow [0, \infty)$ is homeomorphic and depends only on $c_1, c_2, c_3, q_1, q_2, H_1$ and H_2 . Obviously, $g \circ f$ is q -locally weakly H -quasisymmetric, where $H = \max\{\eta(1), 1\}$. Also, it follows from (2) in Remark 1.2 that $g \circ f$ is also H -quasiconformal. \square

REFERENCES

1. G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, The boundary correspondence under quasiconformal mappings, *Trans. Amer. Math. Soc.*, **305** (1988), 95–1111.
2. Z. M. BALOGH AND S. M. BUCKLEY, Sphericalization and flattening, *Conform. Geom. Dyn.*, **9** (2005), 76–101 (electronic).
3. A. BEURLING AND L. V. AHLFORS, The boundary correspondence under quasiconformal mappings, *Acta Math.*, **96** (1956), 125–142.
4. M. BONK, J. HEINONEN, AND P. KOSKELA, Uniformizing Gromov hyperbolic spaces, *Astérisque*, **270** (2001), viii+99 pp.
5. S. BUCKLEY, D. A. HERRON AND X. XIE, Metric space inversions, quasihyperbolic distance, and uniform spaces, *Indiana Univ. Math. J.*, **57** (2008), 837–890.
6. S. BUCKLEY, P. KOSKELA AND G. LU, Boman equals John, *XVIIth Rolf Nevanlinna Colloquium (Joensuu, 1995)*, 91–99, *de Gruyter, Berlin*, 1996.
7. F. W. GEHRING AND B. G. OSGOOD, Uniform domains and the quasi-hyperbolic metric, *J. Analyse Math.*, **36** (1979), 50–74.

8. F. W. GEHRING AND B. P. PALKA, Quasiconformally homogeneous domains, *J. Analyse Math.*, **30** (1976), 172–199.
9. J. HEINONEN, *Lectures on analysis on metric spaces*, Springer, 2001.
10. J. HEINONEN AND P. KOSKELA, Definitions of quasiconformality, *Invent. Math.*, **120** (1995), 61–79.
11. J. HEINONEN AND P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.*, **181** (1998), 1–61.
12. D. A. HERRON, Quasiconformal deformations and volume growth, *Proc. Lond. Math. Soc.*, **92** (2006), 161–199.
13. D. A. HERRON, N. SHANMUGALINGAM AND X. XIE, Uniformity from Gromov hyperbolicity, *Illinois J. Math.*, **52** (2008), 1065–1109.
14. X. HUANG AND J. LIU, Quasihyperbolic metric and quasisymmetric mappings in metric spaces, *Trans. Amer. Math. Soc.*, **367** (2015), 6225–6246.
15. M. HUANG, Y. LI, S. PONNUSAMY AND X. WANG, The quasiconformal subinvariance property of John domains in \mathbb{R}^n and its applications, *Math. Ann.*, DOI 10.1007/s00208-015-1179-y.
16. M. HUANG, Y. LI, M. VUORINEN AND X. WANG, On quasimöbius maps in real Banach spaces, *Israel J. Math.*, **198** (2013), 467–486.
17. M. HUANG, S. PONNUSAMY, A. RASILA AND X. WANG, On quasisymmetry of quasiconformal mappings and its applications, arXiv: 1207.4360v2 [math. CV], 2013.
18. J. MUNKRES, *Topology*, Prentice Hall, Upper Saddle NJ 07458, 2000.
19. A. RASILA AND J. TALPONEN, Convexity properties of quasihyperbolic balls on Banach spaces, *Ann. Acad. Sci. Fenn. Math.*, **37** (2012), 215–228.
20. A. RASILA AND J. TALPONEN, On Quasihyperbolic Geodesics in Banach Spaces, *Ann. Acad. Sci. Fenn. Math.*, **39** (2014), 163–173.
21. E. SOULTANIS AND M. WILLIAMS, Distortion of quasiconformal maps in terms of the quasihyperbolic metric, *J. Math. Anal. Appl.*, **402** (2013), 626–634.
22. P. TUKIA AND J. VÄISÄLÄ, Quasisymmetric embeddings of metric spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **5** (1980), 97–114.
23. P. TUKIA AND J. VÄISÄLÄ, Lipschitz and quasiconformal approximation and extension, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **6** (1981), 303–342.
24. J. TYSON, Quasiconformality and quasisymmetry in metric measure spaces, *Ann. Acad. Sci. Fenn. Math.*, **23** (1998), 525–548.
25. J. VÄISÄLÄ, Quasiconformal concordance, *Monatsh. Math.*, **107** (1989), 155–168.
26. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. I, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **15** (1990), 355–379.
27. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. II, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **16** (1991), 255–310.
28. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. III, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **17** (1992), 393–408.
29. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. IV, *Analysis and Topology*, 697–717, *World Sci. Publ., River Edge, N. J.*, 1998.
30. J. VÄISÄLÄ, Relatively and inner uniform domains, *Conformal Geom. Dyn.*, **2** (1998), 56–88.
31. J. VÄISÄLÄ, The free quasiworld, freely quasiconformal and related maps in Banach spaces, *Quasiconformal geometry and dynamics (Lublin 1996)*, *Banach Center Publications*, Vol. 48, *Polish Academy of Science, Warsaw*, ed. by Bogdan Bojarski, Julian Lawrynowicz, Olli Martio, Matti Vuorinen and Józef Zajac, 1999, 55–118.
32. J. VÄISÄLÄ, Tangential properties of quasihyperbolic geodesics in Banach spaces, *Ann. Acad. Sci. Fenn. Math.*, **38** (2013), 825–837.

MANZI HUANG, DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA,
HUNAN 410081, PEOPLE'S REPUBLIC OF CHINA

E-mail address: mzhuang79@163.com

ANTTI RASILA, DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA,
HUNAN 410081, PEOPLE'S REPUBLIC OF CHINA, AND, DEPARTMENT OF MATHEMATICS AND
SYSTEMS ANALYSIS, AALTO UNIVERSITY, FI-00076 AALTO, FINLAND

E-mail address: antti.rasila@iki.fi

XIANTAO WANG, DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU 515063,
PEOPLE'S REPUBLIC OF CHINA

E-mail address: xtwang@hunnu.edu.cn

QINGSHAN ZHOU, DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU 515063,
PEOPLE'S REPUBLIC OF CHINA

E-mail address: q476308142@qq.com